Virtual Work

3rd Year

Structural Engineering

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Contents

1. Introduction

1.1 General

Virtual Work is a fundamental theory in the mechanics of bodies. So fundamental in fact, that Newton's 3 equations of equilibrium can be derived from it. Virtual Work provides a basis upon which vectorial mechanics (i.e. Newton's laws) can be linked to the energy methods (i.e. Lagrangian methods) which are the basis for finite element analysis and advanced mechanics of materials.

Virtual Work allows us to solve determinate and indeterminate structures and to calculate their deflections. That is, it can achieve everything that all the other methods together can achieve.

Before starting into Virtual Work there are some background concepts and theories that need to be covered.

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1.2   Background

Strain Energy and Work Done

Strain energy is the amount of energy stored in a structural member due to deformation caused by an external load. For example, consider this simple spring:
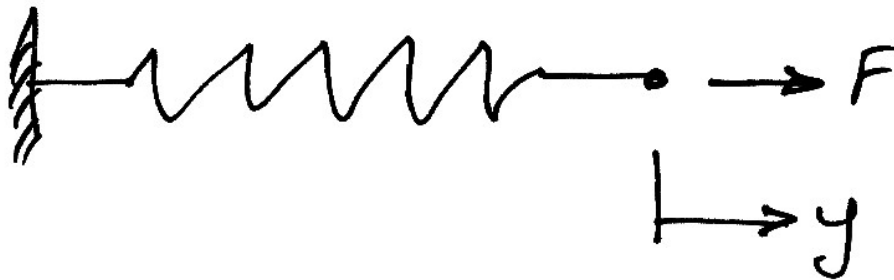
We can see that as it is loaded by a gradually increasing force, *F*, it elongates. We can graph this as:
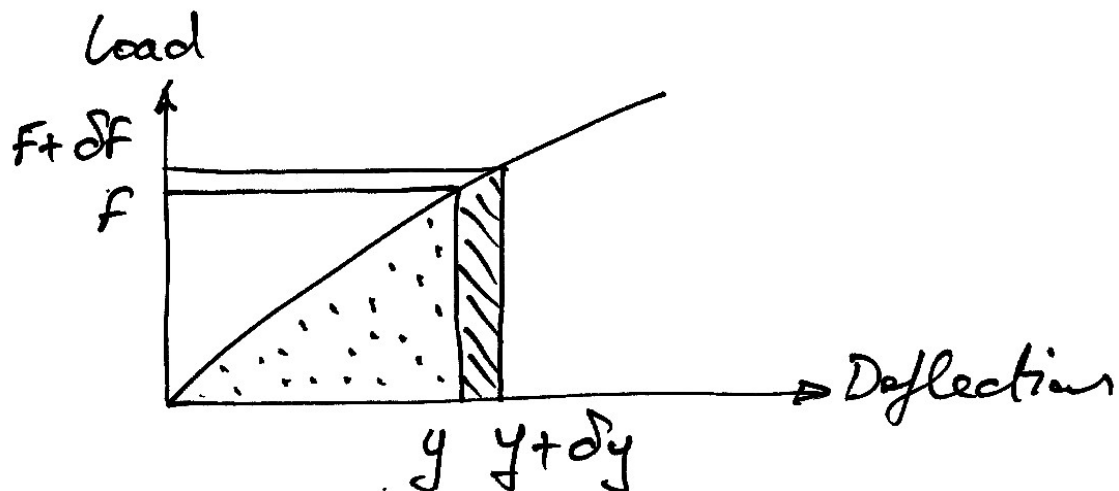
The line OA does not have to be straight, that is, the *constitutive law* of the spring's material does not have to be linear.

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An increase in the force of a small amount, δF results in a small increase in deflection, δy. The work done during this movement (force × displacement) is the average force during the course of the movement, times the displacement undergone. This is the same as the hatched trapezoidal area above. Thus, the increase in work associated with this movement is:

$$\begin{aligned} \delta U &= \frac{F + (F + \delta F)}{2} \cdot \delta y \\ &= F \cdot \delta y + \frac{\delta F \cdot \delta y}{2} \\ &\approx F \cdot \delta y \end{aligned} \qquad (1.1)$$

where we can neglect second-order quantities. As $\delta y \to 0$, we get:

$$dU = F \cdot dy$$

The total work done when a load is gradually applied from 0 up to a force of F is the summation of all such small increases in work, i.e.:

$$U = \int_0^y F \, dy \qquad (1.2)$$

This is the dotted area underneath the load-deflection curve of earlier and represents the work done during the elongation of the spring. This work (or energy, as they are the same thing) is stored in the spring and is called strain energy and denoted U.

If the load-displacement curve is that of a linearly-elastic material then $F = ky$ where k is the constant of proportionality (or the spring stiffness). In this case, the dotted area under the load-deflection curve is a triangle.

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Draw this:
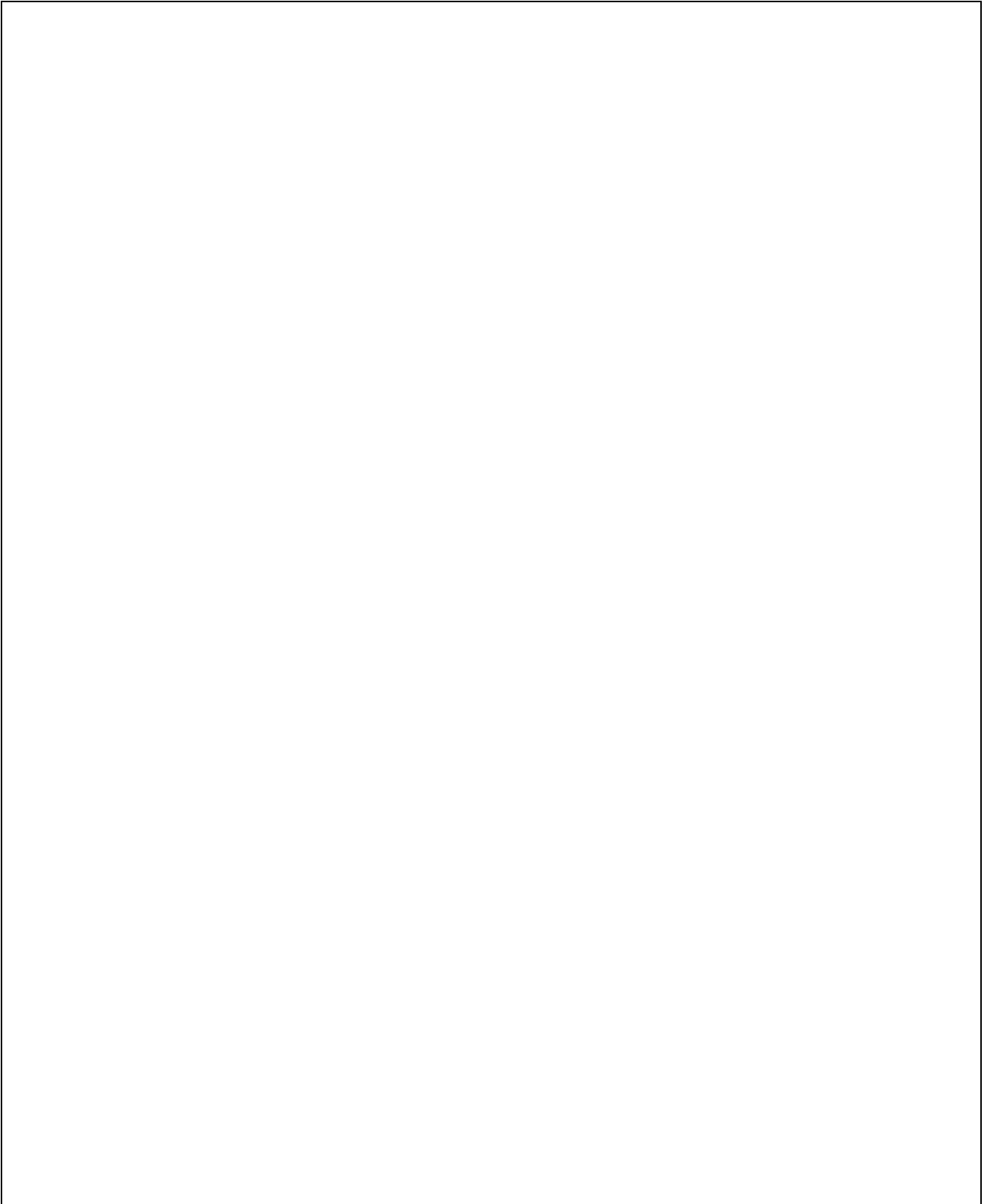
As we know that the work done is the area under this curve, then the work done by the load in moving through the displacement – the *External Work Done*, W_e - is given by:

$$W_e = \frac{1}{2} Fy \qquad (1.3)$$

We can also calculate the strain energy, or *Internal Work Done*, W_I, by:

$$U = \int_0^y F \, dy$$
$$= \int_0^y ky \, dy$$
$$= \frac{1}{2} ky^2$$

Also, since $F = ky$, we then have:

$$W_I = U = \frac{1}{2}(ky)y = \frac{1}{2}Fy$$

But this is the external work done, W_e. Hence we have:

$$W_e = W_I \qquad (1.4)$$

Which we may have expected from the *Law of Conservation of Energy*. Thus:

The external work done by external forces moving through external displacements is equal to the strain energy stored in the material.

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Law of Conservation of Energy

For structural analysis this can be stated as:

Consider a structural system that is isolated such it neither gives nor receives energy; the total energy of this system remains constant.

The isolation of the structure is key: we can consider a structure isolated once we have identified and accounted for all sources of restraint and loading. For example, to neglect the self-weight of a beam is problematic as the beam receives gravitational energy not accounted for, possibly leading to collapse.

In the spring and force example, we have accounted for all restraints and loading (for example we have ignored gravity by having no mass). Thus the total potential energy of the system, Π, is constant both before and after the deformation.

In structural analysis the relevant forms of energy are the potential energy of the load and the strain energy of the material. We usually ignore heat and other energies.
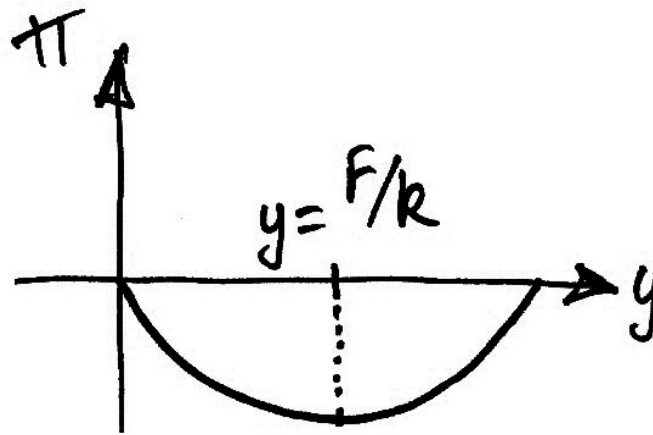
Potential Energy of the Load

Since after the deformation the spring has gained strain energy, the load must have lost potential energy, *V*. Hence, after deformation we have for the total potential energy:

$$\Pi = U + V$$
$$= \frac{1}{2}ky^2 - Fy \tag{1.5}$$

In which the negative sign indicates a loss of potential energy for the load.

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Principle of Minimum Total Potential Energy

If we plot the total potential energy against y, equation (1.5), we get a quadratic curve similar to:

Consider a numerical example, with the following parameters, $k = 10$ kN/m and $F = 10$ kN giving the equilibrium deflection as $y = F/k = 1$ m. We can plot the following quantities:

- Internal Strain Energy, or Internal Work: $U = W_I = \dfrac{1}{2}ky^2 = \dfrac{1}{2}10y^2 = 5y^2$

- Potential Energy: $V = -Fy = -10y$

- Total Potential Energy: $\Pi = U + V = 5y^2 - 10y$

- External Work: $W_e = \dfrac{1}{2}Py = \dfrac{1}{2}10y = 5y$

and we get the following plots (split into two for clarity):

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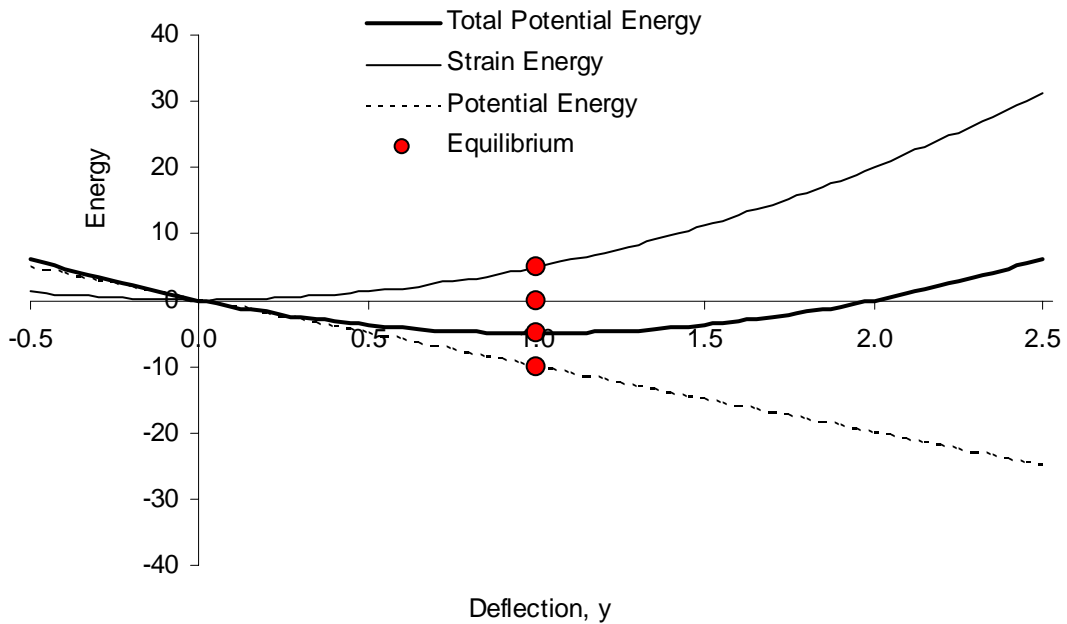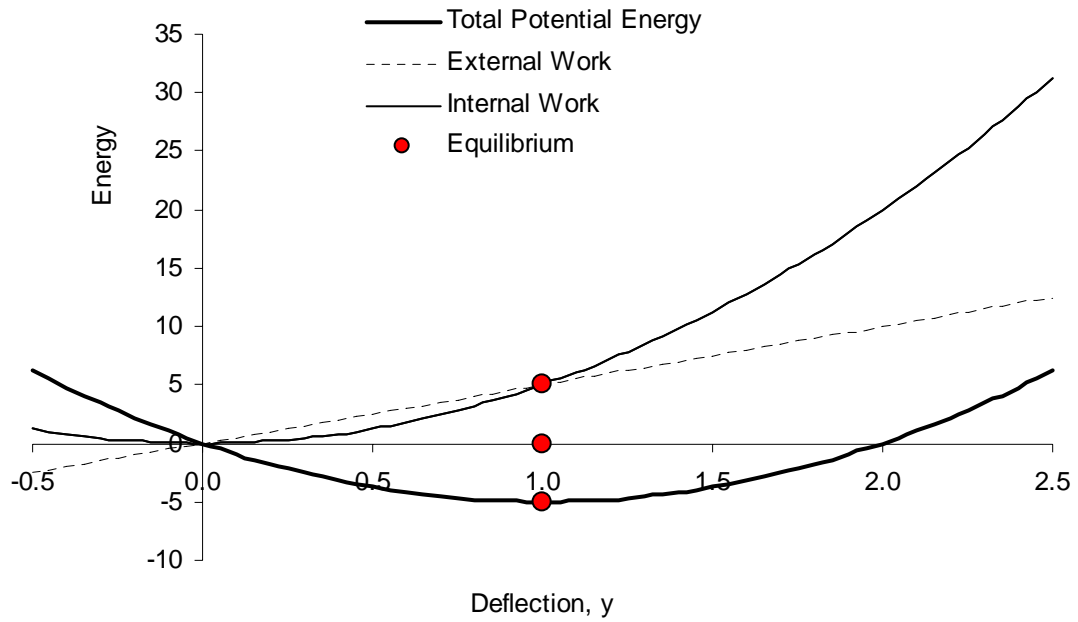
From these graphs we can see that because W_I increases quadratically with y, while the W_e increases only linearly, W_I always catches up with W_e, and there will always be a non-zero equilibrium point where $W_e = W_I$.

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Admittedly, these plots are mathematical: the deflection of the spring will not take up any value; it takes that value which achieves equilibrium. At this point we consider a small *variation* in the total potential energy of the system. Considering F and k to be constant, we can only alter y. The effect of this small variation in y is:

$$
\begin{aligned}
\Pi(y+\delta y)-\Pi(y) &= \frac{1}{2}k(y+\delta y)^2 - F(y+\delta y) - \frac{1}{2}ky^2 + Fy \\
&= \frac{1}{2}k(2y\cdot\delta y) - F\cdot\delta y + \frac{1}{2}k(\delta y)^2 \\
&= (ky-F)\delta y + \frac{1}{2}k(\delta y)^2
\end{aligned}
\tag{1.6}
$$

Similarly to a first derivate, for Π to be an extreme (either maximum or minimum), the first variation must vanish:

$$
\delta^{(1)}\Pi = (ky-F)\delta y = 0
\tag{1.7}
$$

Therefore:

$$
ky - F = 0
\tag{1.8}
$$

Which we recognize to be the $\sum F_x = 0$. Thus equilibrium occurs when Π is an extreme.

Before introducing more complicating maths, an example of the above variation in equilibrium position is the following. Think of a shopkeeper testing an old type of scales for balance – she slightly lifts one side, and if it returns to position, and no large rotations occur, she concludes the scales is in balance. She has imposed a

variation in displacement, and finds that since no further displacement occurs, the 'structure' was originally in equilibrium.

Examining the second variation (similar to a second derivate):

$$\delta^{(2)}\Pi = \frac{1}{2}k(\delta y)^2 \geq 0 \tag{1.9}$$

We can see it is always positive and hence the extreme found was a minimum. This is a particular proof of the general principle that *structures take up deformations that minimize the total potential energy to achieve equilibrium*. In short, nature is lazy!

To summarize our findings:
- Every isolated structure has a total potential energy;
- Equilibrium occurs when structures can minimise this energy;
- A small variation of the total potential energy vanishes when the structure is in equilibrium.

These concepts are brought together in the Principle of Virtual Work.

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2. The Principle of Virtual Work

2.1 Definition

Based upon the Principle of Minimum Total Potential Energy, we can see that any small variation about equilibrium must do no work. Thus, the Principle of Virtual Work states that:

> *A body is in equilibrium if, and only if, the virtual work of all forces acting on the body is zero.*

In this context, the word 'virtual' means 'having the effect of, but not the actual form of, what is specified'. Thus we can imagine ways in which to impose virtual work, without worrying about how it might be achieved in the physical world.

Virtual Work

There are two ways to define virtual work, as follows.

1. Principle of Virtual Displacements:

   Virtual work is the work done by the actual forces acting on the body moving through a virtual displacement.
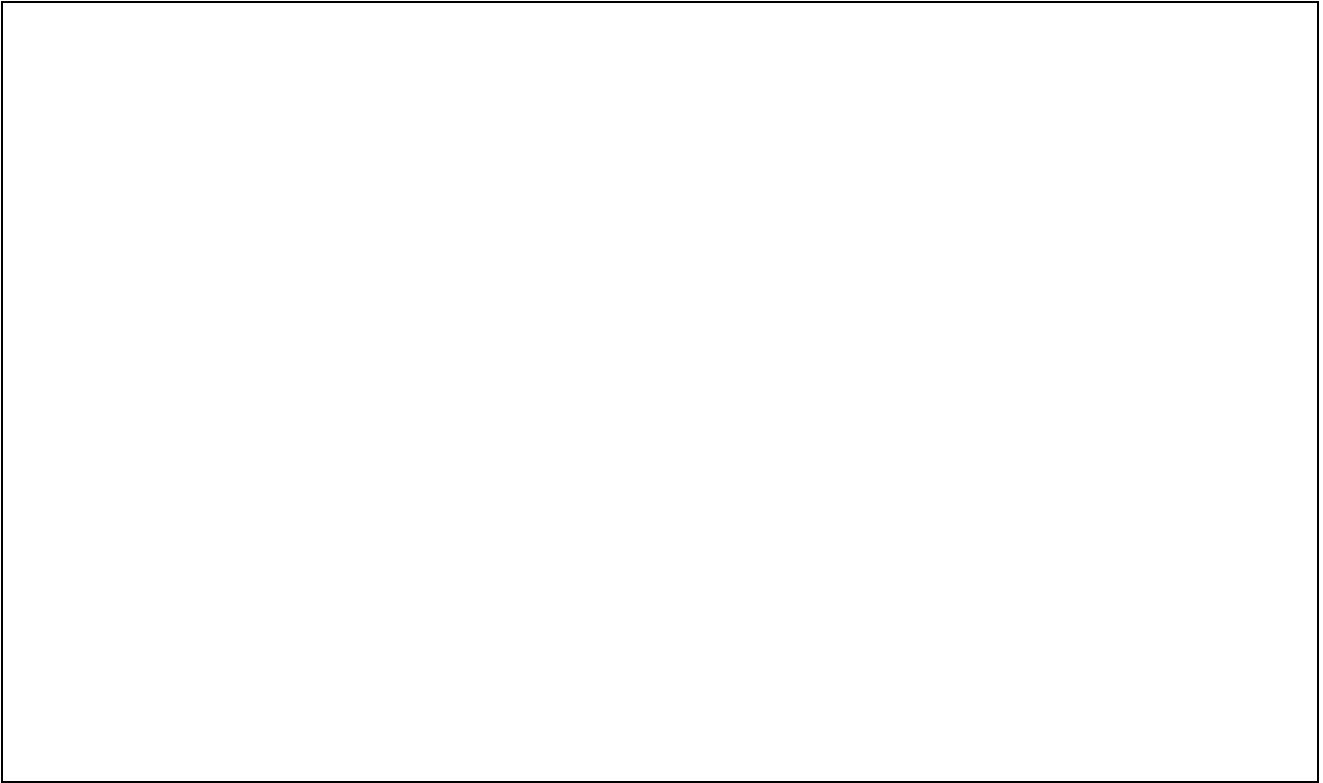
   This means we solve an equilibrium problem through geometry.

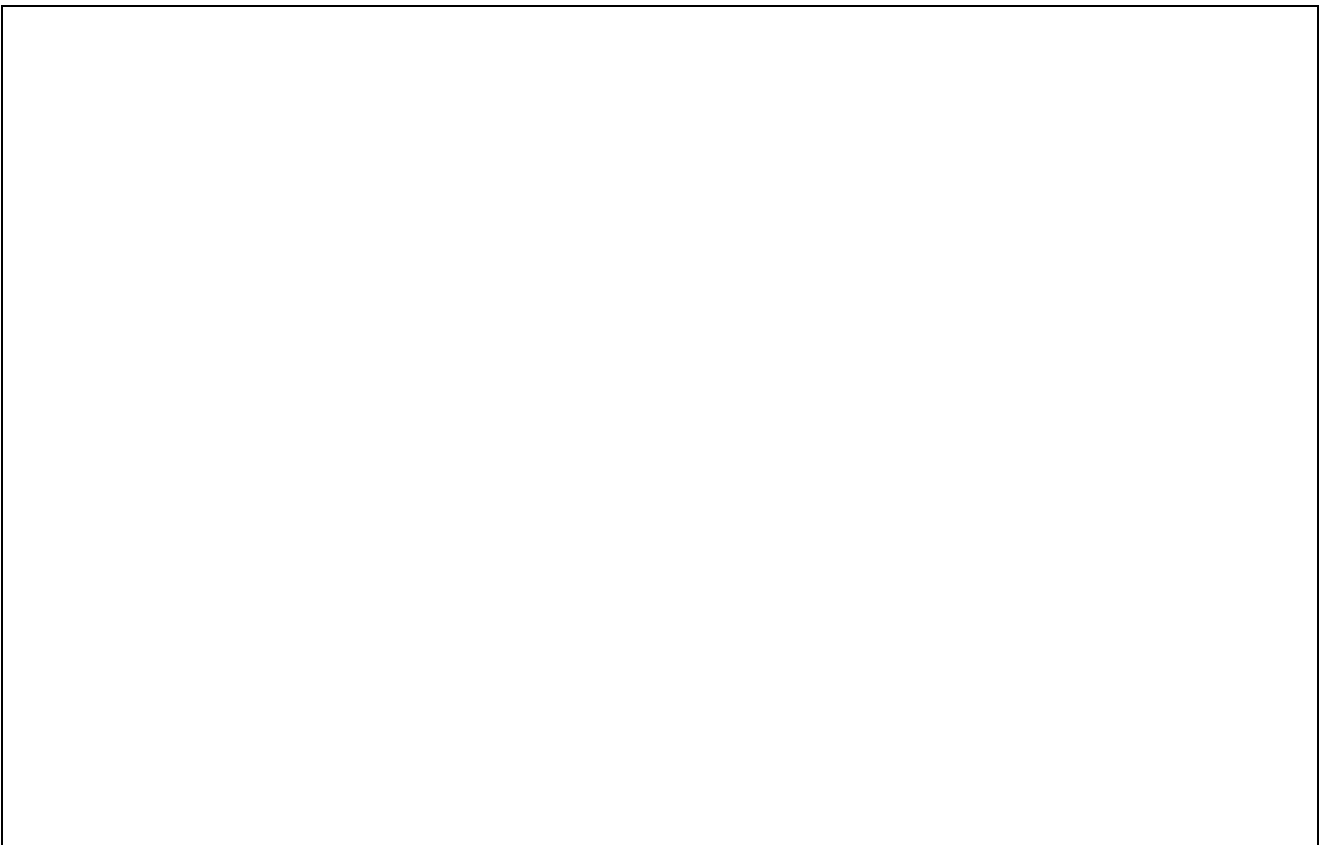2. Principle of Virtual Forces:

   Virtual work is the work done by a virtual force acting on the body moving through the actual displacements.

   This means we solve a geometry problem through equilibrium.

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Illustrate the Principle of Virtual Displacements:

Illustrate the Principle of Virtual Forces:

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2.2 Virtual Displacements

A virtual displacement is a displacement that is only imagined to occur. This is exactly what we did when we considered the vanishing of the first variation of Π; we found equilibrium. Thus the application of a virtual displacement is a means to find this first variation of Π.

So given any *real* force, *F*, acting on a body to which we apply a virtual displacement. If the virtual displacement at the location of and in the direction of *F* is δy, then the force *F* does virtual work $\delta W = F \cdot \delta y$.

There are requirements on what is permissible as a virtual displacement. For example, in the simple proof of the Principle of Virtual Work (to follow) it can be seen that it is assumed that the directions of the forces applied to *P* remain unchanged. Thus:

- virtual displacements must be small enough such that the force directions are maintained.

The other very important requirement is that of compatibility:

- virtual displacements within a body must be geometrically compatible with the original structure. That is, geometrical constraints (i.e. supports) and member continuity must be maintained.

In summary, virtual displacements are not real, they can be physically impossible but they must be compatible with the geometry of the original structure and they must be small enough so that the original geometry is not significantly altered.

As the deflections usually encountered in structures do not change the overall geometry of the structure, this requirement does not cause problems.

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2.3 Virtual Forces

So far we have only considered small virtual displacements and real forces. The virtual displacements are arbitrary: they have no relation to the forces in the system, or its actual deformations. Therefore virtual work applies to any set of forces in equilibrium and to any set of compatible displacements and we are not restricted to considering only real force systems and virtual displacements. Hence, we can use a virtual force system and real displacements. Obviously, in structural design it is these real displacements that are of interest and so virtual forces are used often.

A virtual force is a force imagined to be applied and is then moved through the actual deformations of the body, thus causing virtual work.

So if at a particular location of a structure, we have a deflection, y, and impose a virtual force at the same location and in the same direction of δF we then have the virtual work $\delta W = y \cdot \delta F$.

Virtual forces must form an equilibrium set of their own. For example, if a virtual force is applied to the end of a spring there will be virtual stresses in the spring as well as a virtual reaction.

2.4 Simple Proof using Virtual Displacements

We can prove the Principle of Virtual Work quite simply, as follows. Consider a particle P under the influence of a number of forces $F_1, \ldots, F_n$ which have a resultant force, F_R. Apply a virtual displacement of δy to P, moving it to P', as shown:
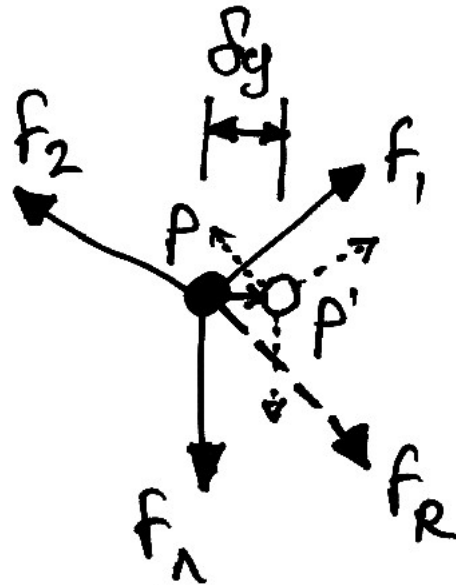
The virtual work done by each of the forces is:

$$\delta W = F_1 \cdot \delta y_1 + \ldots + F_n \cdot \delta y_n = F_R \cdot \delta y_R$$

Where δy_1 is the virtual displacement along the line of action of F_1 and so on. Now if the particle P is in equilibrium, then the forces $F_1, \ldots, F_n$ have no resultant. That is, there is no net force. Hence we have:

$$\delta W = 0 \cdot \delta y_R = F_1 \cdot \delta y_1 + \ldots + F_n \cdot \delta y_n = 0 \tag{2.1}$$

Proving that when a particle is in equilibrium the virtual work of all the forces acting on it sum to zero. Conversely, a particle is only in equilibrium if the virtual work done during a virtual displacement is zero.

2.5   Internal and External Virtual Work

Consider the spring we started with, as shown again below. Firstly it is unloaded. Then a load, F, is applied, causing a deflection y. It is now in its equilibrium position and we apply a virtual displacement, δy to the end of the spring, as shown:
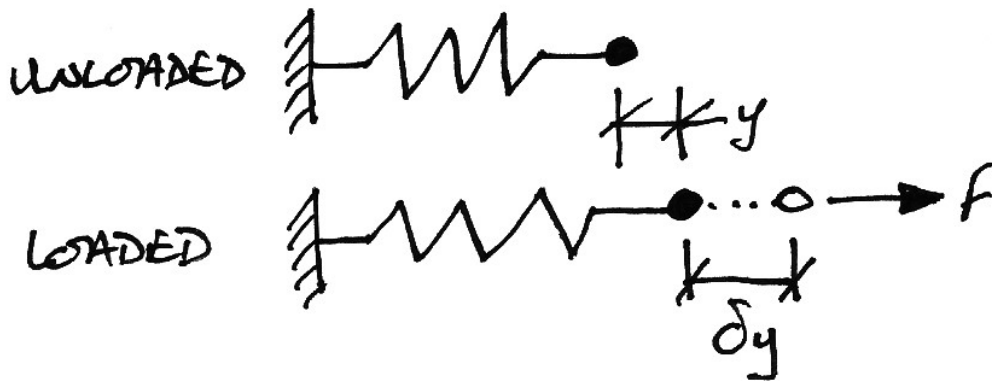
A free body diagram of the end of the spring is:
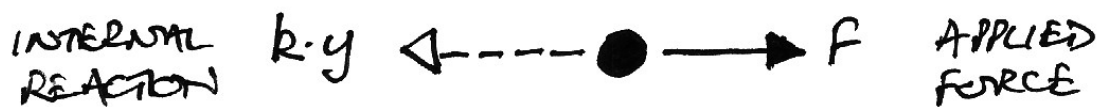
Thus the virtual work done by the two forces acting on the end of the spring is:

$$\delta W = F \cdot \delta y - ky \cdot \delta y$$

If the spring is to be in equilibrium we must then have:

$$\delta W = 0$$
$$F \cdot \delta y - ky \cdot \delta y = 0$$
$$F \cdot \delta y = ky \cdot \delta y$$
$$F = ky$$

That is, the force in the spring must equal the applied force, as we already know. However, if we define the following:

- External virtual work, $\delta W_E = F \cdot \delta y$;

- Internal virtual work, $\delta W_I = ky \cdot \delta y$;

We then have:

$$\delta W = 0$$
$$\delta W_I - \delta W_E = 0$$

Thus:

$$\delta W_E = \delta W_I \qquad (2.2)$$

which states that the external virtual work must equal the internal virtual work for a structure to be in equilibrium.

It is in this form that the Principle of Virtual Work finds most use.

Of significance also in the equating of internal and external virtual work, is that there are no requirements for the material to have any particular behaviour. That is, virtual work applies to all bodies, whether linearly-elastic, elastic, elasto-plastic, plastic etc. Thus the principle has more general application than most other methods of analysis.

Internal and external virtual work can arise from either virtual displacements or virtual forces.

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3. Application of Virtual Displacements

3.1 Rigid Bodies

Basis

Rigid bodies do not deform and so there is no internal virtual work done. Thus:

$$\delta W = 0$$
$$\delta W_E = \delta W_I \tag{3.1}$$
$$\sum_i F_i \cdot \delta y_i = 0$$

A simple application is to find the reactions of statically determinate structures. However, to do so, we need to make use of the following principle:

Principle of Substitution of Constraints

Having to keep the constraints in place is a limitation of virtual work. However, we can substitute the restraining force (i.e. the reaction) in place of the restraint itself. That is, we are turning a geometric constraint into a force constraint. This is the Principle of Substitution of Constraints. We can use this principle to calculate unknown reactions:

1. Replace the reaction with its appropriate force in the same direction (or sense);
2. Impose a virtual displacement on the structure;
3. Calculate the reaction, knowing $\delta W = 0$.

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## Reactions of Determinate and Indeterminate Structures

For statically determinate structures, removing a restraint will always render a mechanism (or rigid body) and so the reactions of statically determinate structures are easily obtained using virtual work. For indeterminate structures, removing a restraint does not leave a mechanism and hence the virtual displacements are harder to establish since the body is not rigid.

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Example 1

Determine the reactions for the following beam:
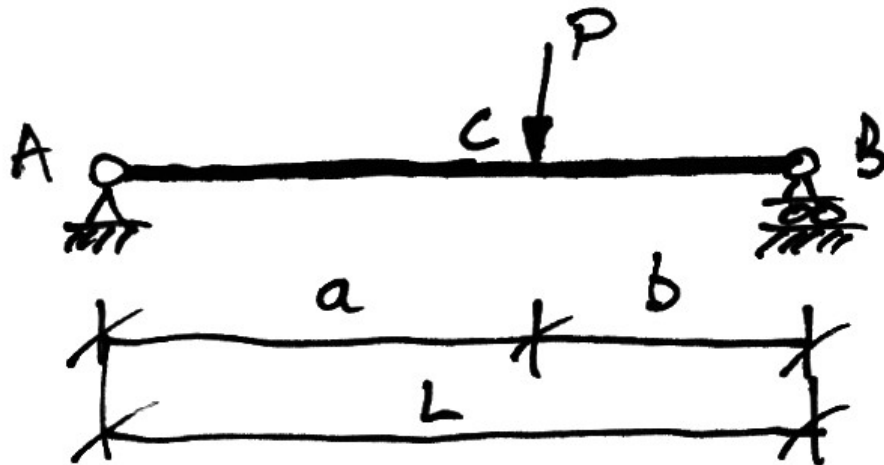
Following the Principle of Substitution of Constraints, we replace the geometric constraints (i.e. supports), by their force counterparts (i.e. reactions) to get the free-body-diagram of the whole beam:
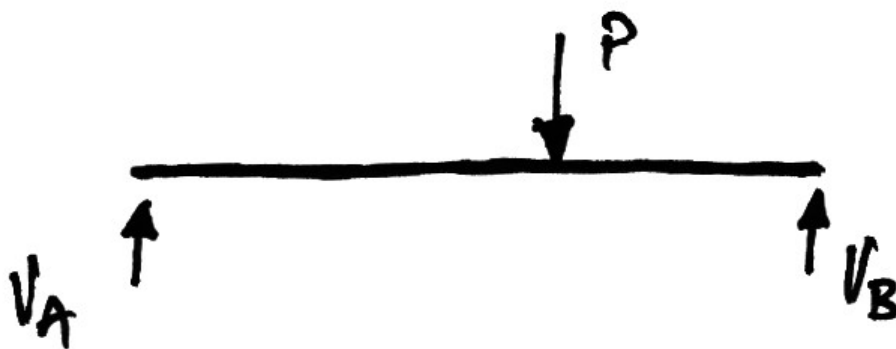
Next, we impose a virtual displacement on the beam. Note that the displacement is completely arbitrary, and the beam remains rigid throughout:

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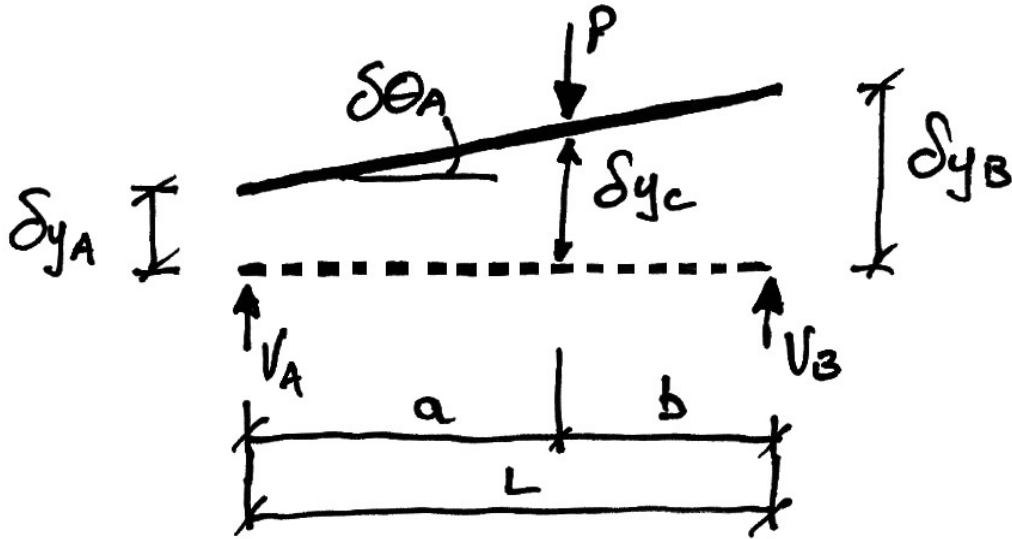
In the above figure, we have imposed a virtual displacement of δy_A at A and then imposed a virtual rotation of $\delta\theta_A$ about A. The equation of virtual work is:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\sum F_i \cdot \delta y_i = 0$$

Hence:

$$V_A \cdot \delta y_A - P \cdot \delta y_C + V_B \cdot \delta y_B = 0$$

Relating the virtual movements of points B and C to the virtual displacements gives:

$$V_A \cdot \delta y_A - P(\delta y_A + a \cdot \delta\theta_A) + V_B(\delta y_A + L \cdot \delta\theta_A) = 0$$

And rearranging gives:

$$(V_A + V_B - P)\delta y_A + (V_B L - Pa)\delta\theta_A = 0$$

And here is the power of virtual work: since we are free to choose any value we want for the virtual displacements (i.e. they are completely arbitrary), we can choose $\delta\theta_A = 0$ and $\delta y_A = 0$, which gives the following two equations:

$$\left(V_A + V_B - P\right)\delta y_A = 0 \qquad\qquad \left(V_B L - Pa\right)\delta\theta_A = 0$$
$$V_A + V_B - P = 0 \qquad\qquad V_B L - Pa = 0$$

But the first equation is just $\sum F_y = 0$ whilst the second is the same as $\sum M$ about $A = 0$. Thus equilibrium equations occur naturally within the virtual work framework. Note also that the two choices made for the virtual displacements correspond to the following virtual displaced configurations: **Draw them**
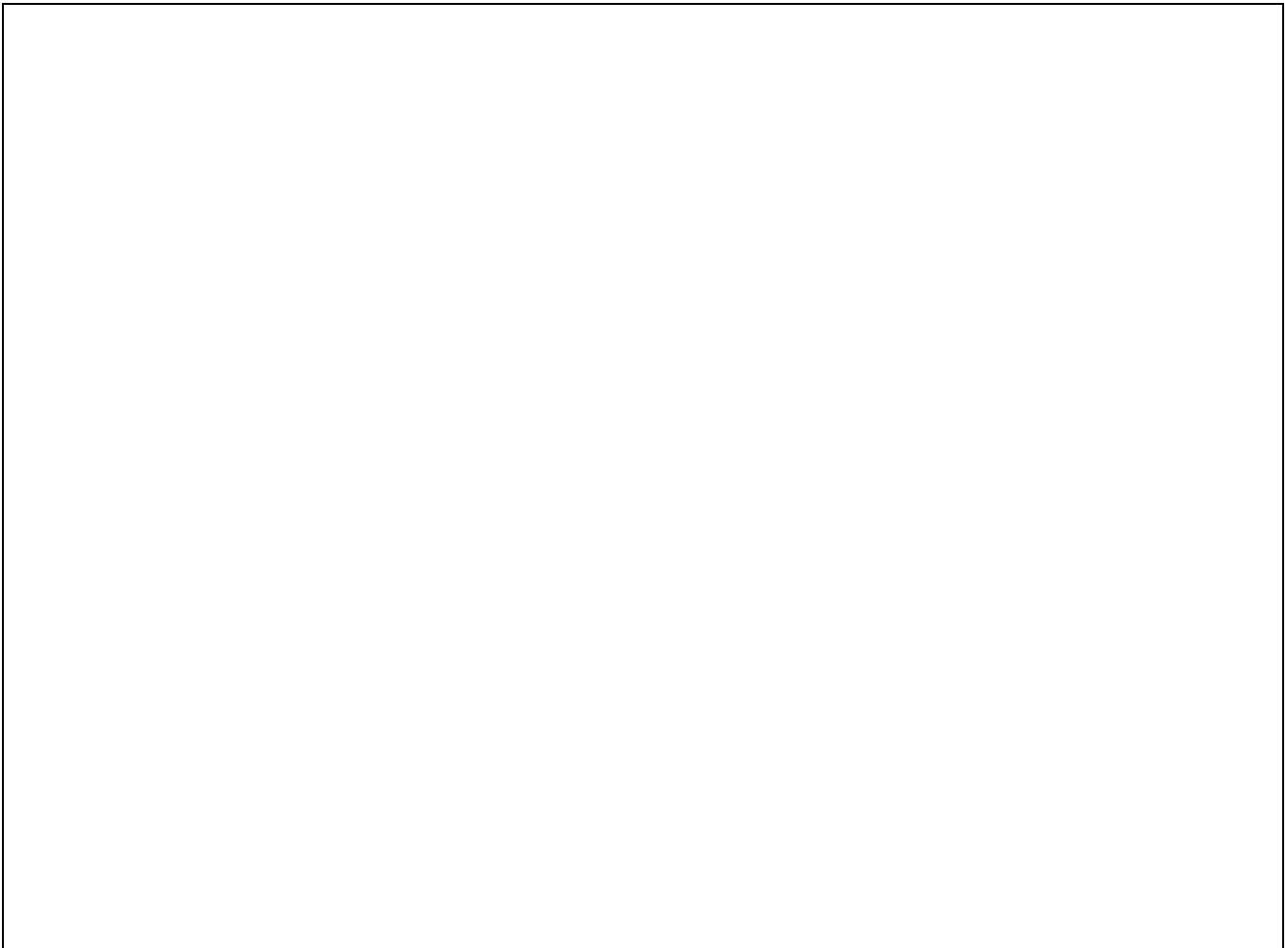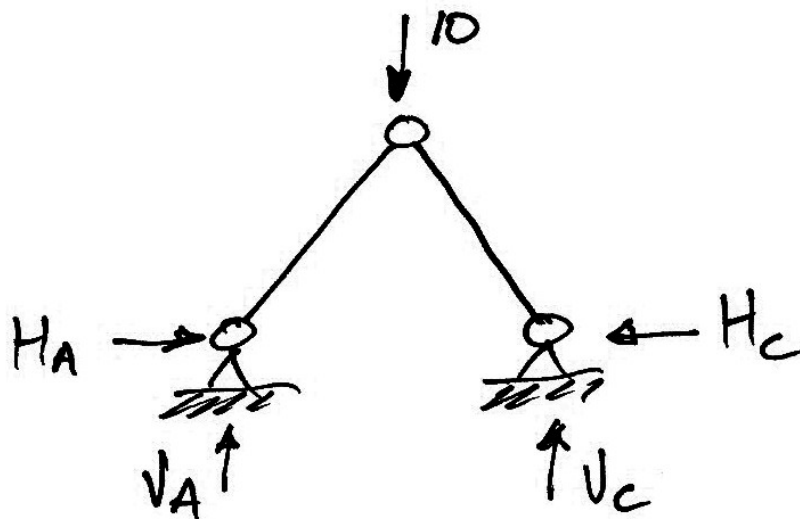
Example 2
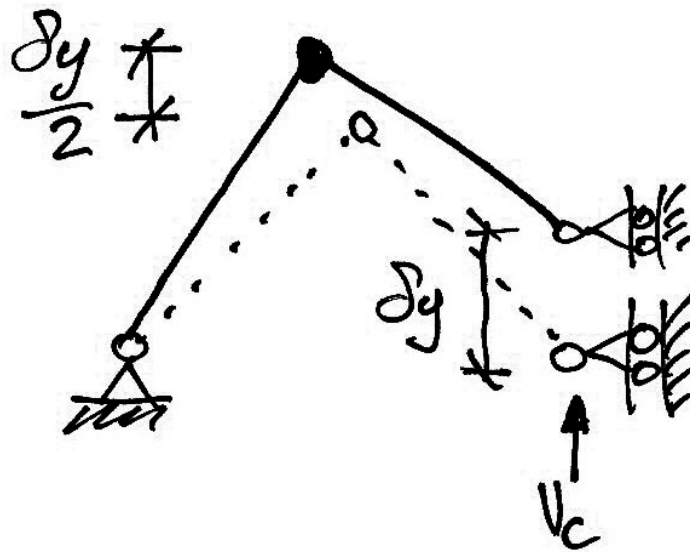
For the following truss, calculate the reaction V_C:
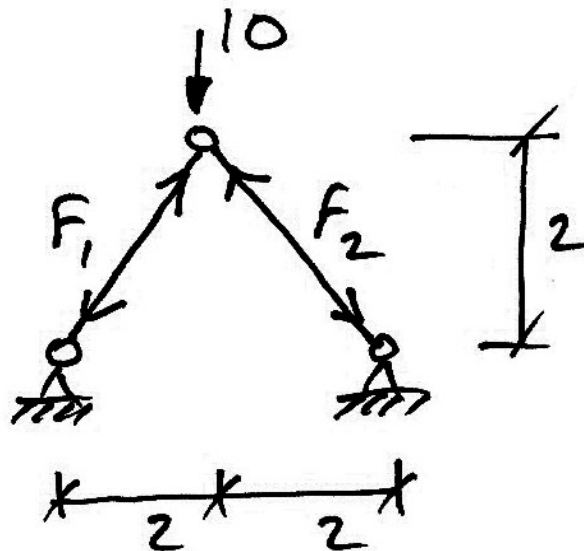
Firstly, set up the free-body-diagram of the whole truss:

Next we release the constraint corresponding to reaction V_C and replace it by the unknown force V_C and we apply a virtual displacement to the truss to get:

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Hence the virtual work done is:

$$\delta W = 0$$

$$\delta W_E = \delta W_I$$

$$-10 \cdot \frac{\delta y}{2} + V_C \cdot \delta y = 0$$

$$V_C = 5 \text{ kN}$$

Note that the reaction is an external force to the structure, and that no internal virtual work is done since the members do not undergo virtual deformation. The truss rotates as a rigid body about the support A.

Problem:

Find the horizontal reactions of the truss.

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3.2   Deformable Bodies

Basis

For a virtual displacement we have:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\sum F_i \cdot \delta y_i = \int P_i \cdot \delta e_i$$

In which, for the external virtual work, F_i represents an externally applied force (or moment) and δy_i its virtual displacement. And for the internal virtual work, P_i represents the internal force in member i and δe_i its virtual deformation. Different stress resultants have different forms of internal work, and we will examine these. Lastly, the summations reflect the fact that all work done must be accounted for.

Remember in the above, each the displacements must be compatible and the forces must be in equilibrium, summarized as:
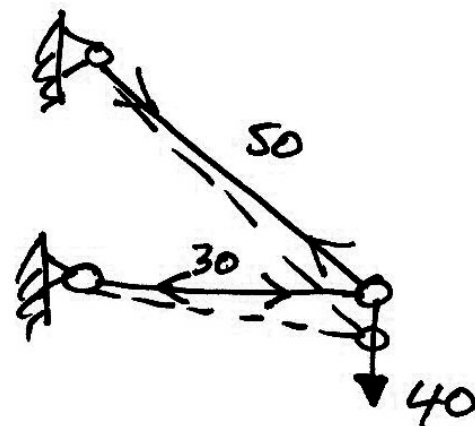
Set of forces in

equilibrium

$$\sum F_i \cdot \delta y_i = \sum P_i \cdot \delta e_i$$

Set of compatible
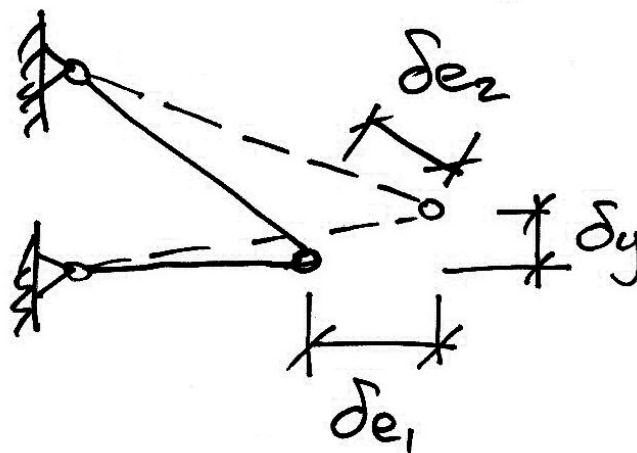
displacements

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These displacements are completely arbitrary (i.e. we can choose them to suit our purpose) and bear no relation to the forces above. For example, the simple truss shown has a set of forces in equilibrium along with its actual deformed shape. Also shown is an arbitrary set of permissible compatible displacements:
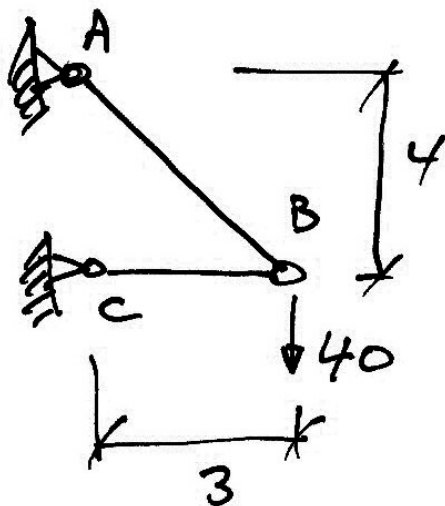
Loading and dimensions   **Equilibrium forces and actual deformation**

Compatible set of displacements

Internal Virtual Work by Axial Force

Members subject to axial force may have the following:

- real force by a virtual displacement:

$$\delta W_I = P \cdot \delta e$$

- virtual force by a real displacement:

$$\delta W_I = e \cdot \delta P$$

We have avoided integrals over the length of the member since we will only consider prismatic members.

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Internal Virtual Work in Bending

The internal virtual work done in bending is one of:

- real moment by a virtual curvature:

$$\delta W_I = M \cdot \delta \kappa$$

- virtual moment by a real curvature:

$$\delta W_I = \kappa \cdot \delta M$$

The above expressions are valid at a single position in a beam.

When virtual rotations are required along the length of the beam, the easiest way to do this is by applying virtual loads that in turn cause virtual moments and hence virtual curvatures. We must sum all of the real moments by virtual curvatures along the length of the beam, hence we have:

$$\delta W_I = \int_0^L M_x \cdot \delta \kappa_x \, dx = \int_0^L \kappa_x \cdot \delta M_x \, dx$$

$$= \int_0^L M_x \cdot \frac{\delta M_x}{EI_x} \, dx$$

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Internal Virtual Work in Shear

At a single point in a beam, the shear strain, γ, is given by:

$$\gamma = \frac{V}{GA_v}$$

where V is the applied shear force, G is the shear modulus and A_v is the cross-section area effective in shear which is explained below. The internal virtual work done in shear is thus:

- real shear force by a virtual shear strain:

$$\delta W_I = V \cdot \delta\gamma = V \cdot \frac{\delta V}{GA_v}$$

- virtual shear force by a real shear strain:

$$\delta W_I = \gamma \cdot \delta V = \frac{V}{GA_v} \cdot \delta V$$

In these expressions, the area of cross section effective in shear comes about because the shear stress (and hence strain) is not constant over a cross section. The stress V/A_v is an average stress, equivalent in work terms to the actual uneven stress over the full cross section, A. We say $A_v = A/k$ where k is the shear factor for the cross section. Some values of k are 1.2 for rectangular sections, 1.1 for circular sections, and 2.0 for thin-walled circular sections.

The above expressions are valid at a single position in a beam and must be integrated along the length of the member as was done for moments and rotations.

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Internal Virtual Work in Torsion

At a single point in a member, the twist, ϕ, is given by:

$$\phi = \frac{T}{GJ}$$

where T is the applied torque, G is the shear modulus, J is the polar moment of inertia. The internal virtual work done in torsion is thus:

- real torque by a virtual twist:

$$\delta W_I = T \cdot \delta\phi = T \cdot \frac{\delta T}{GJ}$$

- virtual torque by a real twist:

$$\delta W_I = \delta T \cdot \phi = \delta T \cdot \frac{T}{GJ}$$

Once again, the above expressions are valid at a single position in a beam and must be integrated along the length of the member as was done for moments and rotations.
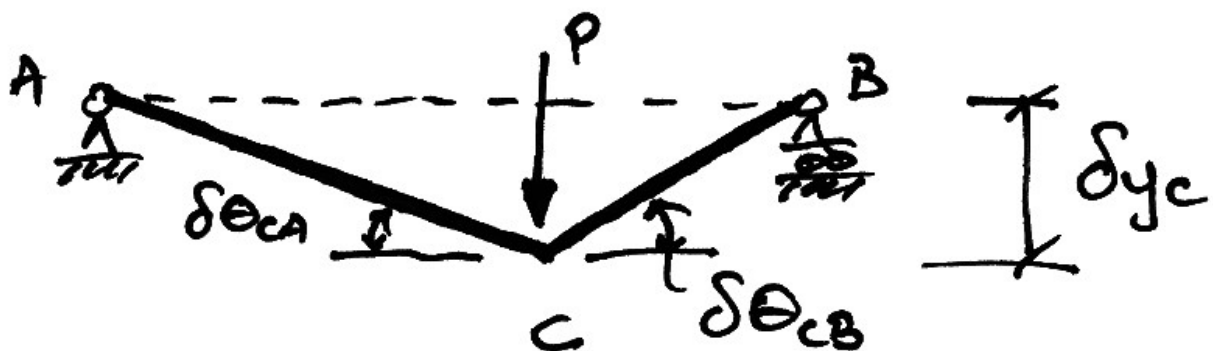
Note the similarity between the expressions for the four internal virtual works.

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Example 3

For the beam of Example 1 (shown again), find the bending moment at *C*.
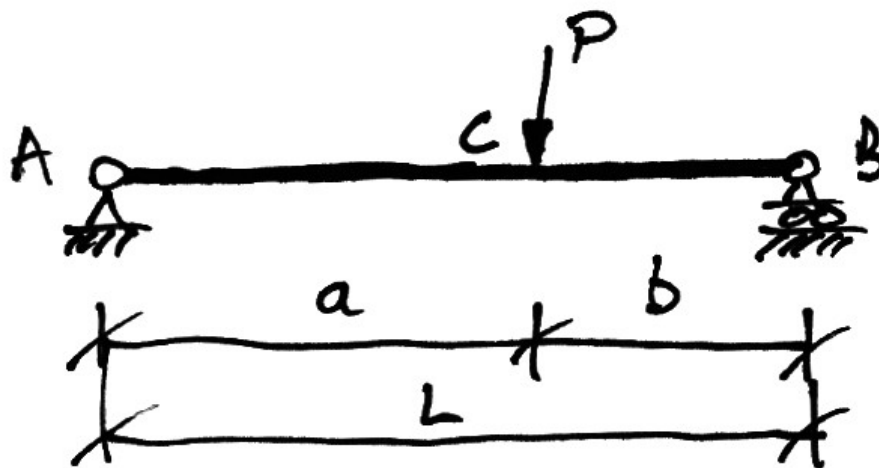
To solve this, we want to impose a virtual displacement configuration that only allows the unknown of interest, i.e. M_C, to do any work. Thus choose the following:

Since portions *AC* and *CB* remain straight (or unbent) no internal virtual work is done in these sections. Thus the only internal work is done at *C* by the beam moving through the rotation at *C*. Thus:

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$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$P \cdot \delta y_C = M_C \cdot \delta \theta_C$$

Bu the rotation at C is made up as:

$$\delta \theta_C = \delta \theta_{CA} + \delta \theta_{CB}$$
$$= \frac{\delta y_C}{a} + \frac{\delta y_C}{b}$$
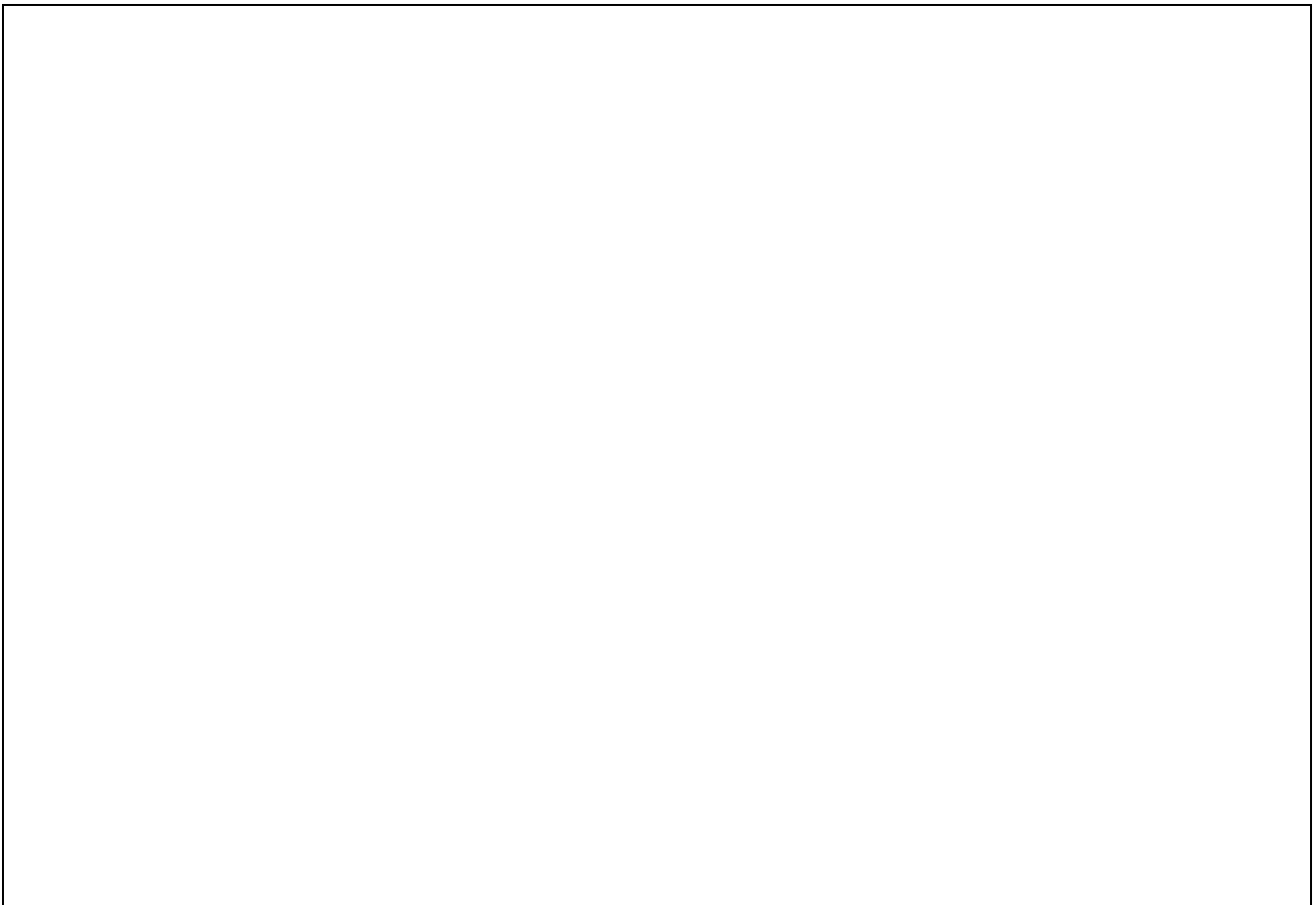$$= \left(\frac{a+b}{ab} \right) \delta y_C$$

But $a + b = L$, hence:

$$P \cdot \delta y_C = M_C \cdot \left(\frac{L}{ab} \right) \delta y_C$$
$$M_C = \frac{Pab}{L}$$

We can verify this from the reactions found previously: $M_C = V_B \cdot b = \left(Pa/L \right) b$.

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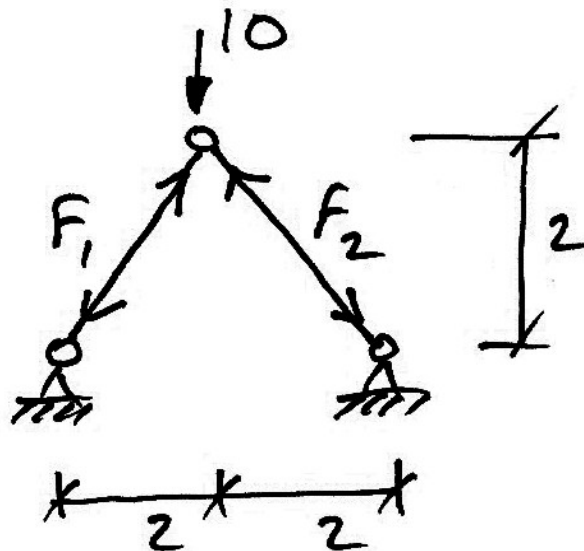
Sign Convention for Rotations

When imposing a virtual rotation, if the side that is already in tension due to the real force system elongates, then positive virtual work is done.

Sketch this:

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Example 4

Calculate the force F_1 in the truss shown:

To do this, we introduce a virtual displacement along the length of member 1. We do this so that member 2 does not change length during this virtual displacement and so does no virtual work. Note also that compatibility of the truss is maintained. For example, the members still meet at the same joint, and the support conditions are met.

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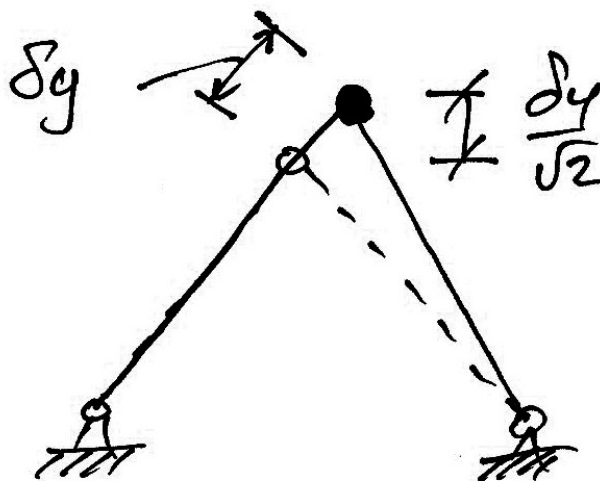
The virtual work done is then:

$$\delta W = 0$$

$$\delta W_E = \delta W_I$$

$$-10 \cdot \frac{\delta y}{\sqrt{2}} = -F_1 \cdot \delta y$$

$$F_1 = \frac{10}{\sqrt{2}} = 5\sqrt{2} \text{ kN}$$

Note some points on the signs used:

1. Negative external work is done because the 10 kN load moves upwards; i.e. the reverse direction to its action.

2. We assumed member 1 to be in compression but then applied a virtual elongation to the member thus reducing its internal virtual work. Hence negative internal work is done.

3. We initially assumed F_1 to be in compression and we obtained a positive answer confirming our assumption.

Problem:

Investigate the vertical and horizontal equilibrium of the loaded joint by considering vertical and horizontal virtual displacements separately.

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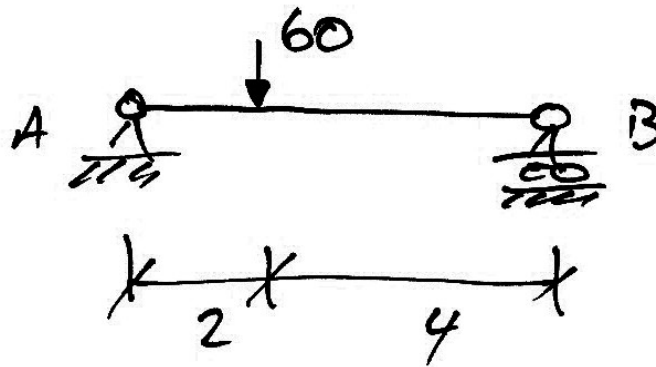
3.3 Problems

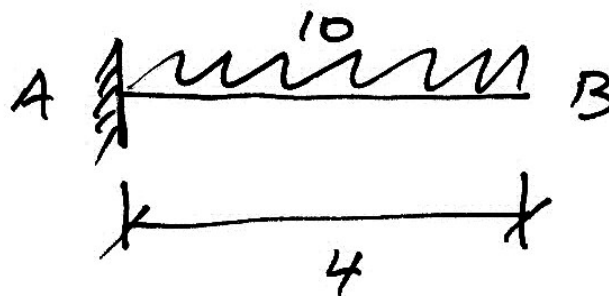1. For the truss shown, calculate the vertical reaction at C and the forces in the members, using virtual work.
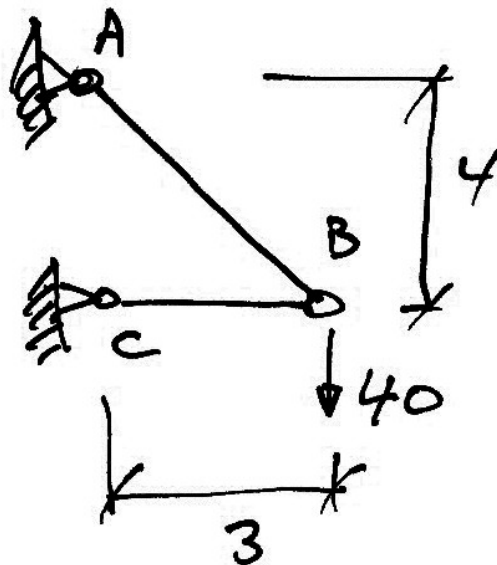
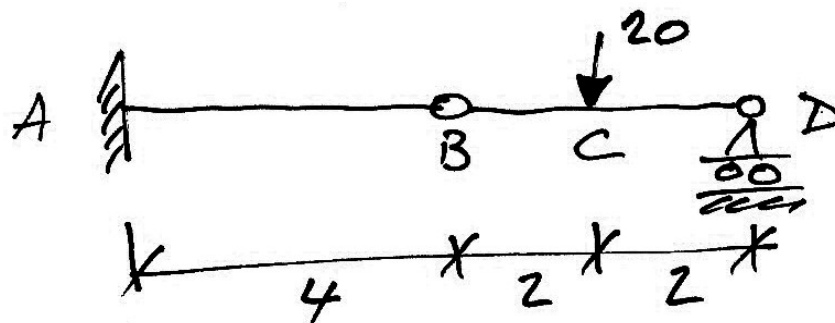2. For the truss shown, find the forces in the members, using virtual work:

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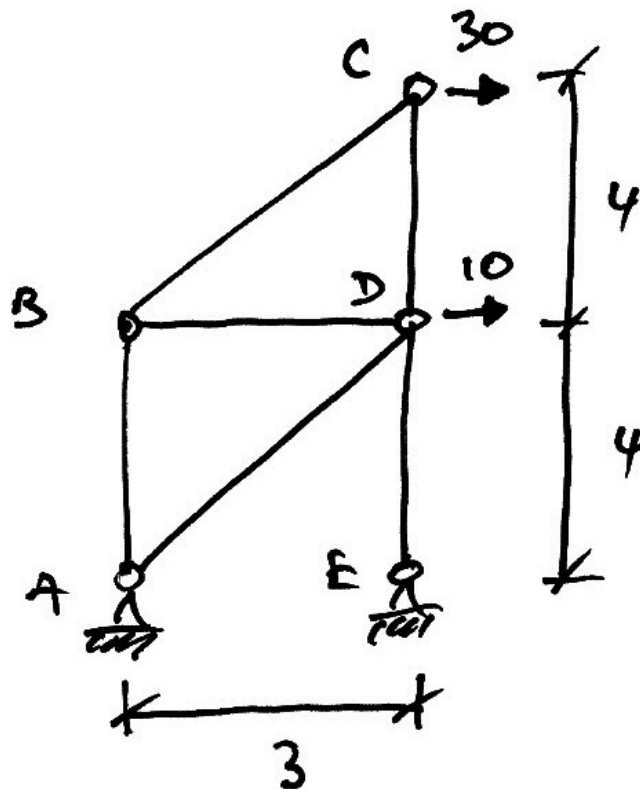
3. Using virtual work, calculate the reactions for the beams shown, and the bending moments at salient points.
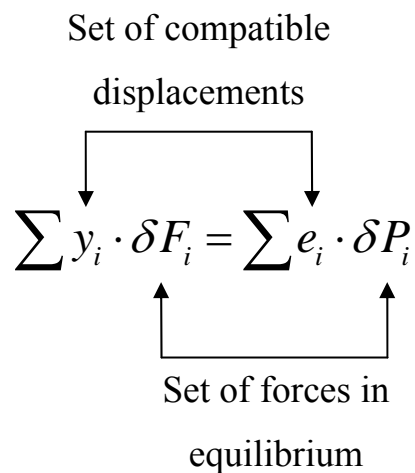

Beam 1

Beam 2

Beam 3

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4. Application of Virtual Forces

4.1 Basis

When virtual forces are applied, we have:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$

And again note that we have an equilibrium set of forces and a compatible set of displacements:

Set of compatible
displacements

$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$

Set of forces in
equilibrium

In this case the displacements are the real displacements that occur when the structure is in equilibrium and the virtual forces are any set of arbitrary forces that are in equilibrium.
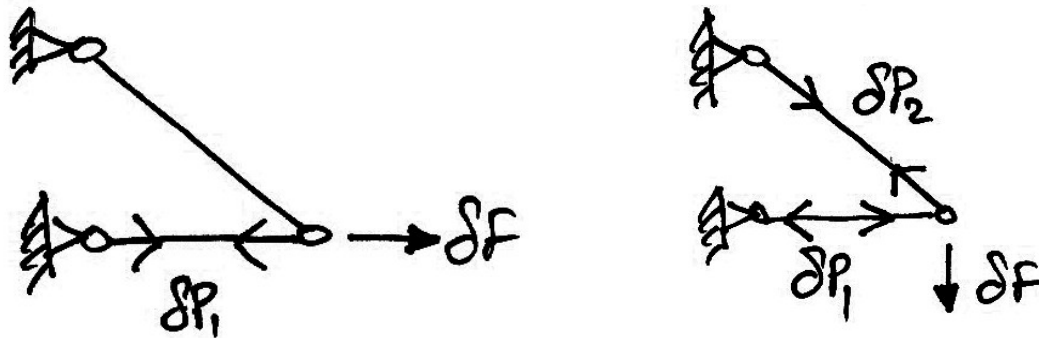
4.2 Deflection of Trusses

Illustrative Example

For the truss shown below, we have the actual displacements shown with two possible sets of virtual forces.
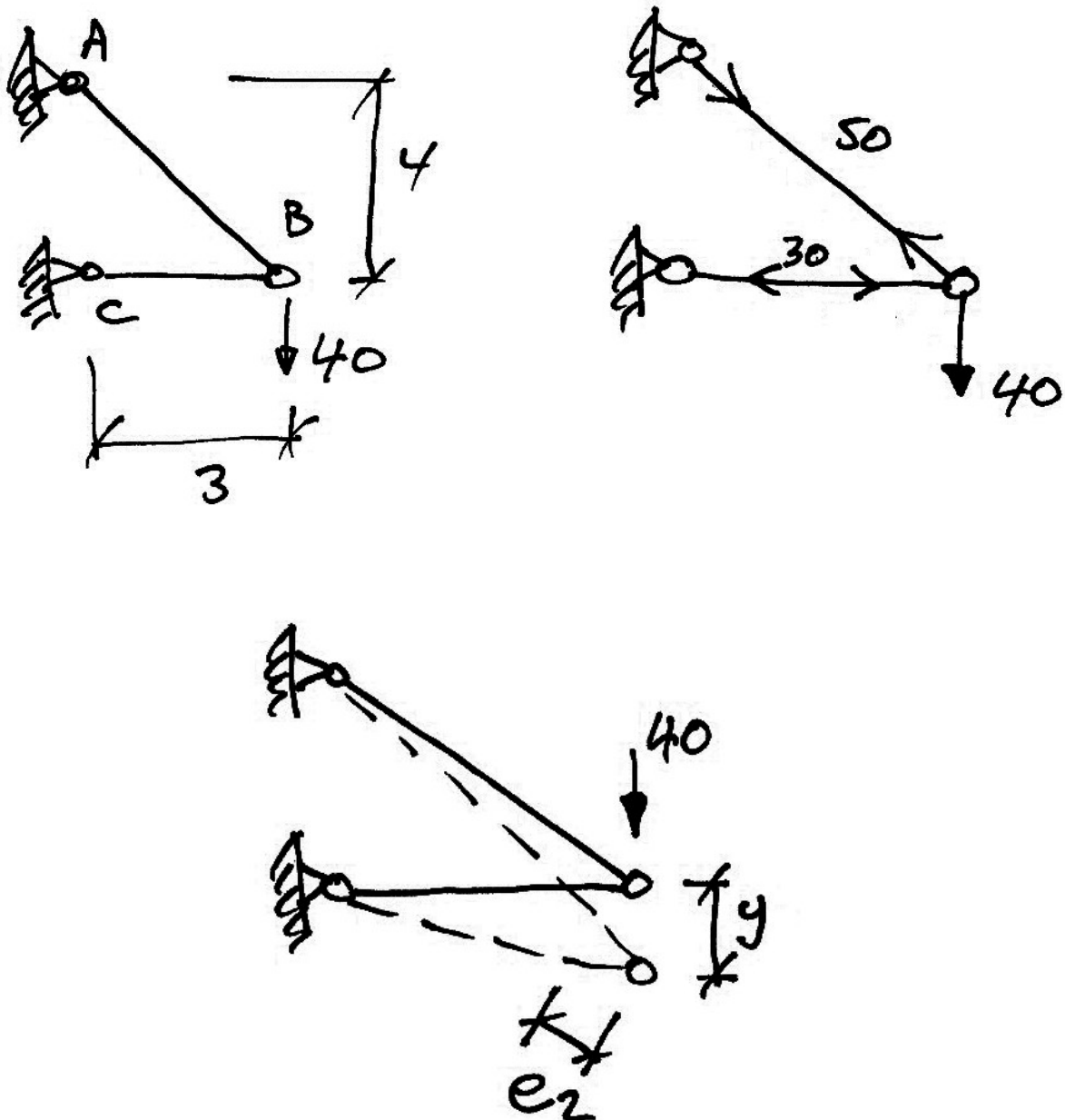
Actual Loading and (Compatible) Displacements

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Virtual Force (Equilibrium) Systems

In this truss, we know we know:

1. The forces in the members (got from virtual displacements or statics);

2. Thus we can calculate the member extensions, e_i as:

$$e_i = \left(\frac{PL}{EA} \right)_i$$
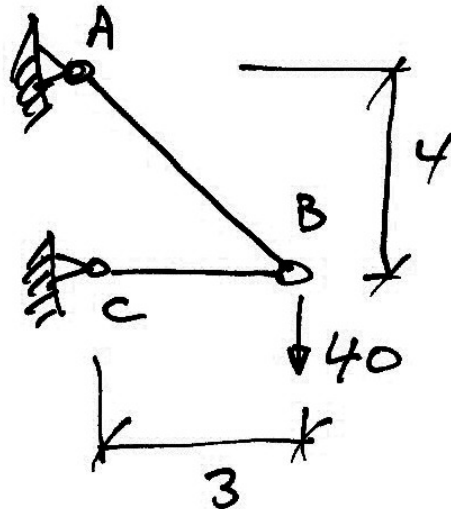
3. Also, as we can choose what our virtual force δF is (usually unity), we have:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$
$$y \cdot \delta F = \sum \left(\frac{PL}{EA} \right)_i \cdot \delta P_i$$

4. Since in this equation, y is the only unknown, we can calculate the deflection of the truss.

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Example 5

Given that $E = 200$ kN/mm^2 and $A = 100$ mm^2 for each member, for the truss shown below, calculate the vertical and horizontal deflection of joint B.

In these problems we will always choose $\delta F = 1$. Hence we apply a unit virtual force to joint B. We apply the force in the direction of the deflection required. In this way no work is done along deflections that are not required. Hence we have:

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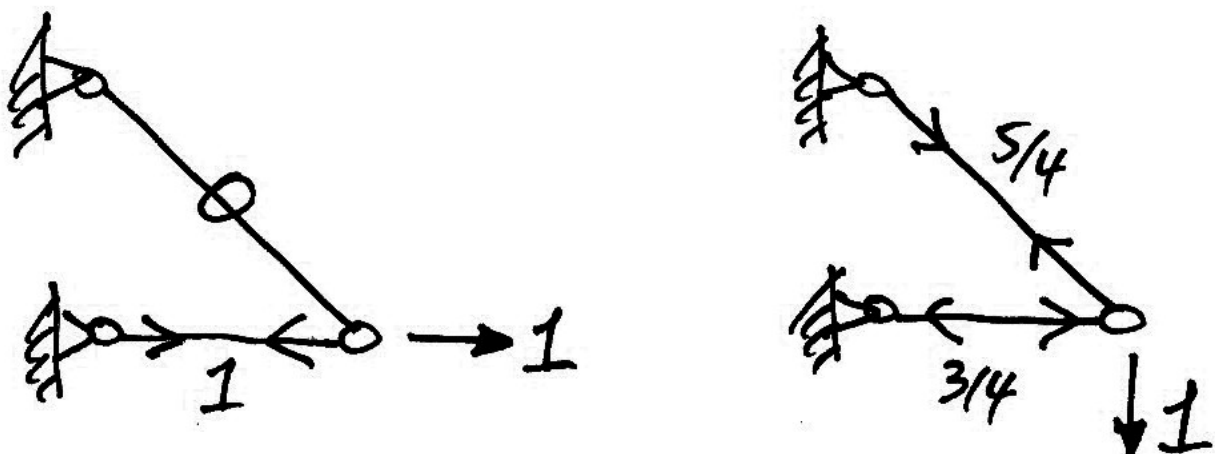
The forces and elongations of the truss members are:

- Member AB:

$$P_{AB} = +50 \text{ kN}$$

$$e_{AB} = \left(\frac{+50 \cdot 5000}{200 \cdot 100} \right)$$

$$= +12.5 \text{ mm}$$

- Member BC:

$$P_{BC} = -30 \text{ kN}$$

$$e_{BC} = \left(\frac{-30 \cdot 3000}{200 \cdot 100} \right)$$

$$= -4.5 \text{ mm}$$

Note that by taking tension to be positive, elongations are positive and contractions are negative.

Horizontal Deflection:

$$\delta W = 0$$

$$\delta W_E = \delta W_I$$

$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$

$$y \cdot 1 = \left(12.5 \cdot 0\right)_{AB} + \left(-4.5 \cdot +1\right)_{BC}$$

$$y = -4.5 \text{ mm}$$

Vertical Deflection:

$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$

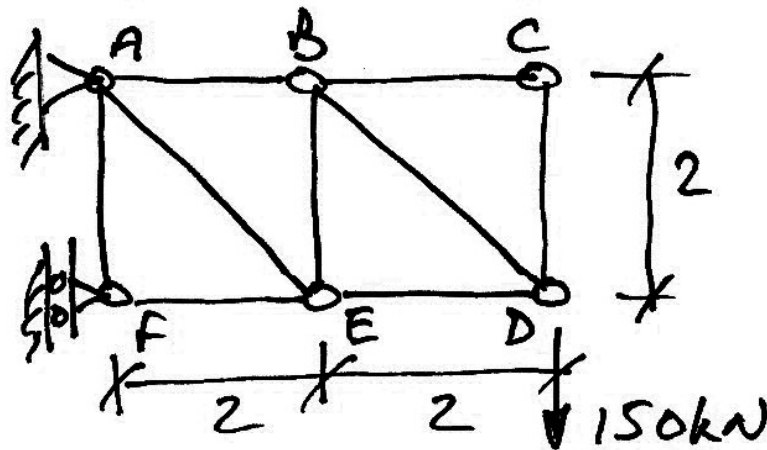$$y \cdot 1 = \left(12.5 \cdot \frac{5}{4}\right)_{AB} + \left(-4.5 \cdot -\frac{3}{4}\right)_{BC}$$

$$y = +18.4 \text{ mm}$$

Note that the sign of the result indicates whether the deflection occurs in the same direction as the applied force. Hence, joint B moves 4.5 mm to the left.

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Example 6

Determine the vertical and deflection of joint *D* of the truss shown. Take $E = 200$ kN/mm^2 and member areas, $A = 1000$ mm^2 for all members except *AE* and *BD* where $A = 1000\sqrt{2}$ mm^2.

The elements of the virtual work equation are:

- Compatible deformations: The actual displacements that the truss undergoes;
- Equilibrium set: The external virtual force applied at the location of the required deflection and the resulting internal member virtual forces.

Firstly we analyse the truss to determine the member forces in order to calculate the actual deformations of each member:

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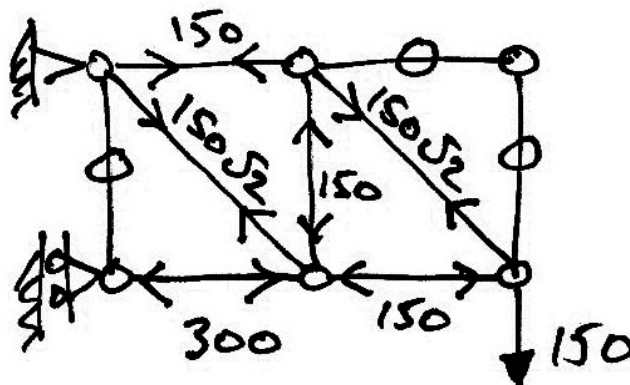
Next, we apply a unit virtual force in the vertical direction at joint D. However, by linear superposition, we know that the internal forces due to a unit load will be 1/150 times those of the 150 kN load.

For the horizontal deflection at D, we apply a unit horizontal virtual force as shown:
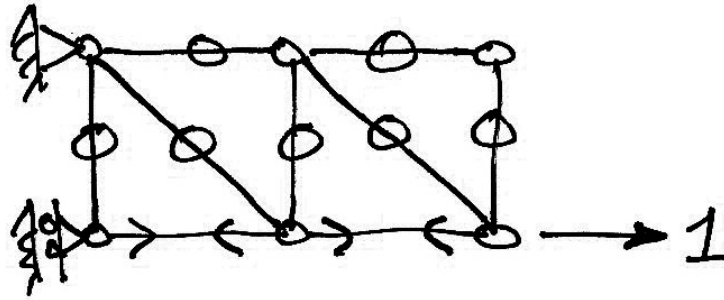
Equations of Virtual Work

$$\delta W = 0$$

$$\delta W_E = \delta W_I$$

$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$

$$y_{DV} \cdot 1 = \sum \left(\frac{P^0 L}{EA} \right)_i \cdot \delta P_i^1$$

$$y_{DH} \cdot 1 = \sum \left(\frac{P^0 L}{EA} \right)_i \cdot \delta P_i^2$$

In which:

- P^0 are the forces due to the 150 kN load;

- δP^1 are the virtual forces due to the unit virtual force applied in the vertical direction at D:

$$\delta P^1 = \frac{P^0}{150}$$

- δP^2 are the virtual forces due to the unit virtual force in the horizontal direction at D.

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Using a table is easiest because of the larger number of members:

Member	L	A	P^0	$\delta P^1 = \dfrac{P^0}{150}$	δP^2	$\left(\dfrac{P^0 L}{A}\right) \cdot \delta P^1$	$\left(\dfrac{P^0 L}{A}\right) \cdot \delta P^2$
	(mm)	(mm^2)	(kN)	(kN)	(kN)	(kN/mm)×kN	(kN/mm)×kN
AB	2000	1000	+150	+1	0	+300	0
AE	2000√2	1000√2	+150√2	+1√2	0	+600	0
AF	2000	1000	0	0	0	0	0
BC	2000	1000	0	0	0	0	0
BD	2000√2	1000√2	+150√2	+1√2	0	+600	0
BE	2000	1000	-150	-1	0	+300	0
CD	2000	1000	0	0	0	0	0
DE	2000	1000	-150	-1	+1	+300	-300
EF	2000	1000	-300	-2	+1	+1200	-600
					$\sum=$	3300	-900

E is left out because it is common. Returning to the equations, we now have:

$$y_{DV} \cdot 1 = \frac{1}{E} \sum \left(\frac{P^0 L}{A}\right)_i \cdot \delta P_i^1$$

$$y_{DV} = \frac{+3300}{200} = +16.5 \text{ mm}$$

Which indicates a downwards deflection and for the horizontal deflection:

$$y_{DH} \cdot 1 = \frac{1}{E} \sum \left(\frac{P^0 L}{A}\right)_i \cdot \delta P_i^2$$
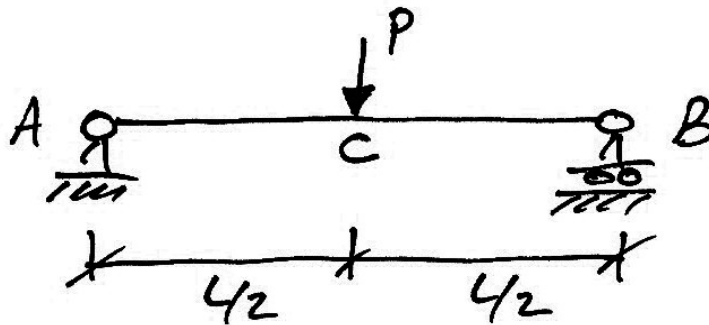
$$y_{DH} = \frac{-900}{200} = -4.5 \text{ mm}$$

The sign indicates that it is deflecting to the left.

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4.3 Deflection of Beams and Frames

Example 7

Using virtual work, calculate the deflection at the centre of the beam shown, given that *EI* is constant.

To calculate the deflection at *C*, we will be using virtual forces. Therefore the two relevant sets are:

- Compatibility set: the actual deflection at *C* and the rotations that occur along the length of the beam;
- Equilibrium set: a unit virtual force applied at *C* which is in equilibrium with the internal virtual moments it causes.

Compatibility Set:

The external deflection at *C* is what is of interest to us. To calculate the rotations along the length of the beam we have:

$$\kappa = \int_0^L \frac{M_x}{EI_x}dx$$

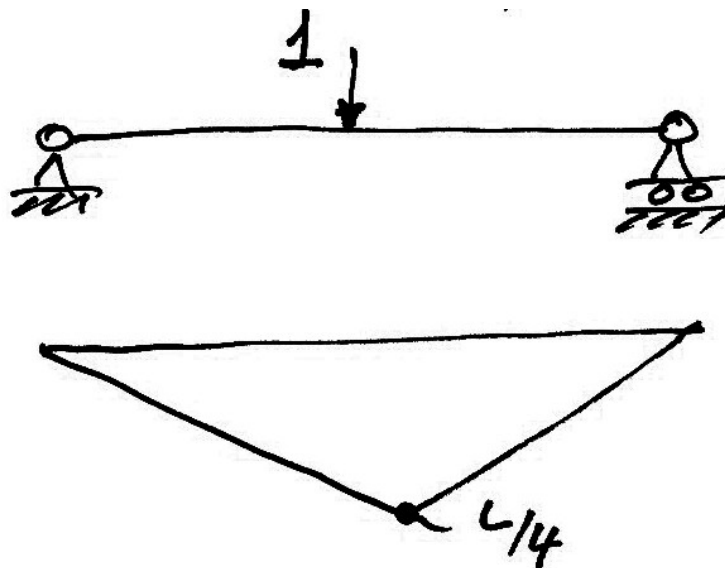Hence we need to establish the bending moments along the beam:

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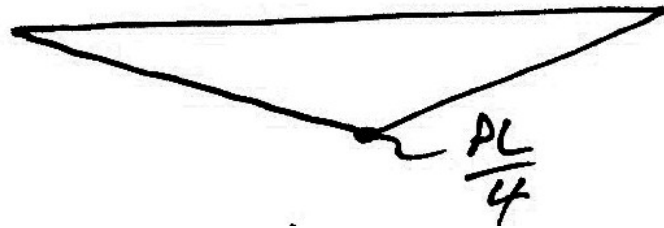
For *AC* the bending moment is given by (and similarly for *B* to *C*):

$$M_x = \frac{P}{2}x$$

Equilibrium Set:

As we choose the value for $\delta F = 1$, we are only left to calculate the virtual moments:

For *AC* the internal virtual moments are given by:

$$\delta M_x = \frac{1}{2}x$$

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Virtual Work Equation

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\sum y_i \cdot \delta F_i = \sum \kappa_i \cdot \delta M_i$$

Substitute in the values we have for the real rotations and the virtual moments, and use the fact that the bending moment diagrams are symmetrical:

$$y \cdot 1 = 2 \int_0^{L/2} \left[\frac{M_x}{EI} \right] \cdot \delta M_x \, dx$$

$$y = \frac{2}{EI} \int_0^{L/2} \left(\frac{P}{2} x \right) \cdot \left(\frac{1}{2} x \right) dx$$

$$= \frac{2P}{4EI} \int_0^{L/2} x^2 \, dx$$

$$= \frac{P}{2EI} \left[\frac{x^3}{3} \right]_0^{L/2}$$

$$= \frac{P}{6EI} \cdot \frac{L^3}{8}$$

$$= \frac{PL^3}{48EI}$$

Which is a result we expected.

Example 8

Find the vertical and horizontal displacement of the prismatic curved bar cantilever shown:
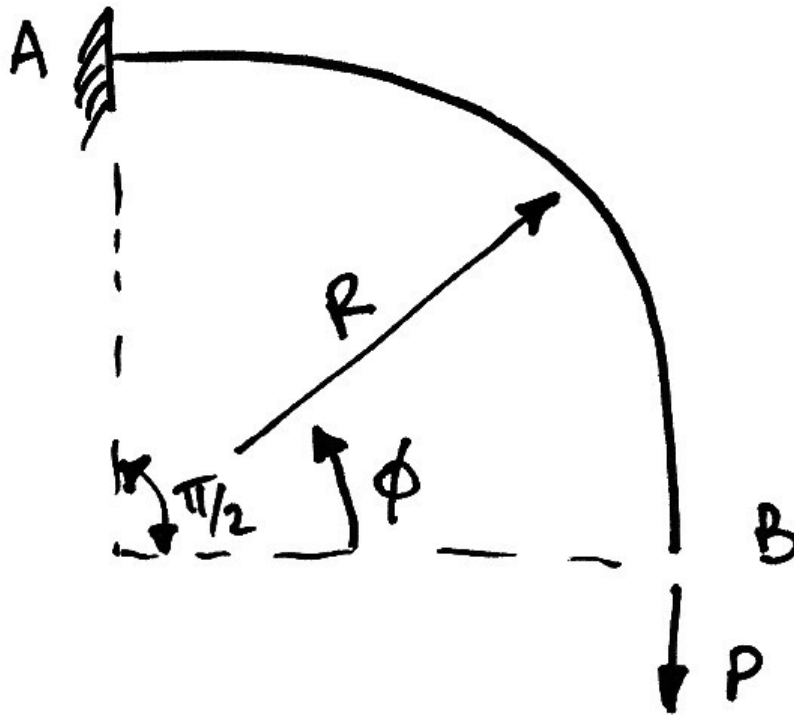
Even though it is curved, from the statics, the bending moment at any point is still force by distance. Hence, at any angle ϕ, we have:
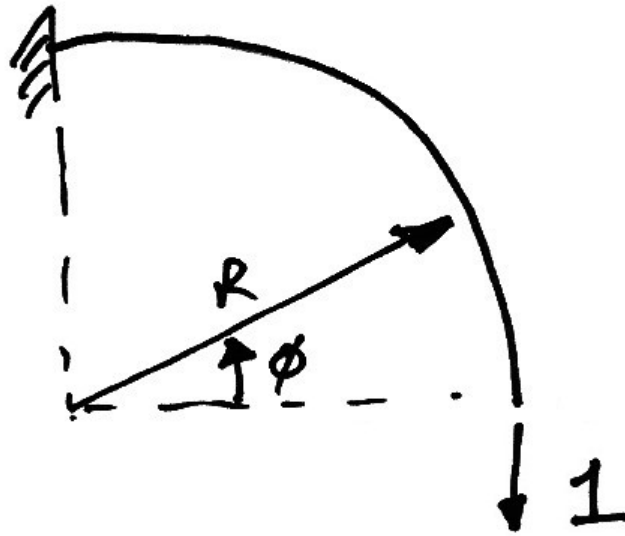
$$
\begin{aligned}
M_\phi &= P(R - R\cos\phi) \\
&= PR(1 - \cos\phi)
\end{aligned}
$$

To find the displacements, we follow our usual procedure and place a unit load at the location of, and in the direction of, the required displacement.

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Vertical Displacement

The virtual bending moment induced by the vertical unit load shown, is related to that for *P* and is thus:

$$\delta M_\phi = R(1 - \cos\phi)$$

Thus our virtual work equations are:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$1 \cdot \delta_{BV} = \int \kappa \cdot \delta M \cdot ds$$
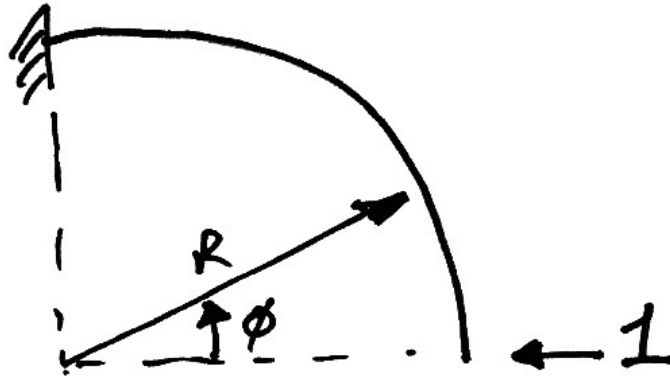$$= \int_0^{\pi/2} \frac{M_\phi}{EI} \cdot \delta M_\phi \cdot R d\phi$$

In which we have used the relation $ds = R\,d\phi$ to change the integration from along the length of the member to around the angle ϕ. Next we introduce our equations for the real and virtual bending moments:

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$$1 \cdot \delta_{BV} = \int_0^{\pi/2} \frac{PR(1-\cos\phi)}{EI} \cdot R(1-\cos\phi) \cdot R d\phi$$

$$= \frac{PR^3}{EI} \int_0^{\pi/2} (1-\cos\phi)^2 \cdot d\phi$$

$$= \frac{PR^3}{EI}(3\pi - 4) \approx 5.42 \frac{PR^3}{EI}$$

Horizontal Displacement

In this case, the virtual bending moment is:

$$\delta M_\phi = R\sin\phi$$

Thus the virtual work equations give:

$$1 \cdot \delta_{BH} = \int_0^{\pi/2} \frac{PR(1-\cos\phi)}{EI} \cdot R\sin\phi \cdot R d\phi$$

$$= \frac{PR^3}{EI} \int_0^{\pi/2} (\sin\phi - \sin\phi\cos\phi) \cdot d\phi$$

$$= \frac{PR^3}{EI} \int_0^{\pi/2} \left(\sin\phi - \frac{\sin 2\phi}{2} \right) \cdot d\phi$$

$$= \frac{PR^3}{EI} \left(\frac{1}{2} \right) = \frac{PR^3}{2EI}$$

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4.4 Integration of Bending Moments

We are often faced with the integration of being moment diagrams when using virtual work to calculate the deflections of bending members. And as bending moment diagrams only have a limited number of shapes, a table of 'volume' integrals is used:

This table is at the back page of these notes for ease of reference.

Example 9

Using the table of volume integrals, verify the answer to Example 7.

In this case, the virtual work equation becomes:

$$y \cdot 1 = 2 \int_0^{L/2} \left[\frac{M_x}{EI} \right] \cdot \delta M_x \, dx$$

$$y = \frac{2}{EI} [M_x \text{ shape}] \times [\delta M_x \text{ shape}]$$

$$= \frac{2}{EI} \left[\frac{1}{3} \cdot \frac{PL}{4} \cdot \frac{L}{4} \cdot \frac{L}{2} \right]$$

$$= \frac{PL^3}{48EI}$$

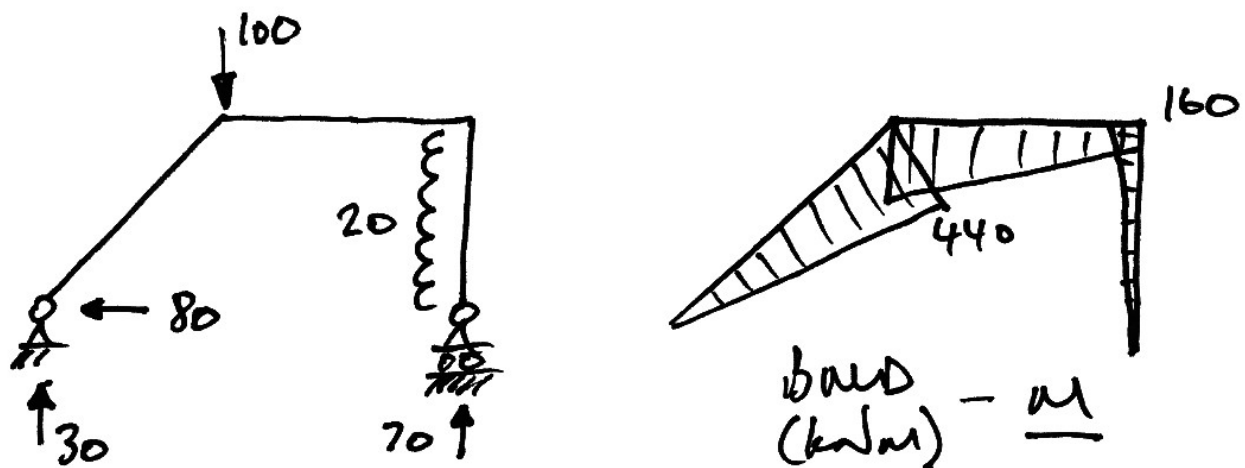In which the formula $\frac{1}{3} jkl$ is used from the table.

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Example 10 – Summer '07 Part (a)

For the frame shown, determine the horizontal deflection of joint C. Neglect axial effects in the members and take $EI = 36 \times 10^3$ kNm^2.
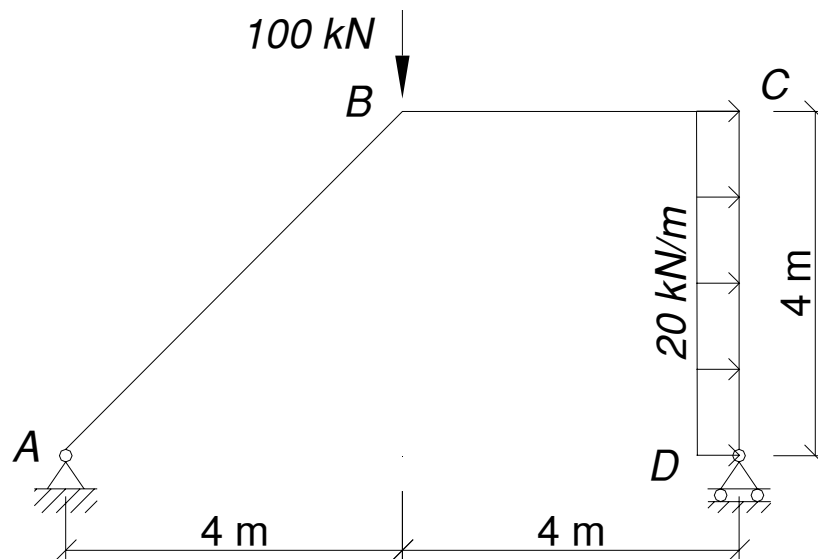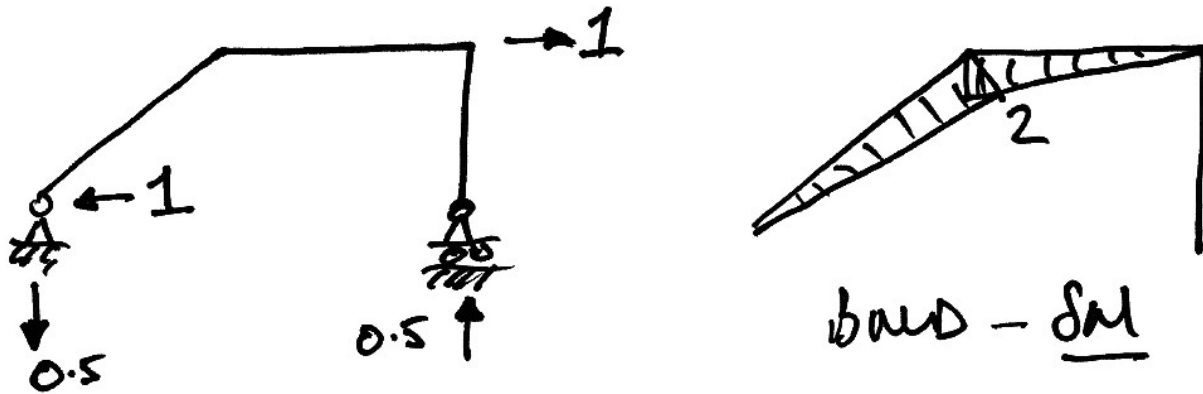

Firstly we establish the real bending moment diagram:

Next, as usual, we place a unit load at the location of, and in the direction of, the required displacement:

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Now we have the following for the virtual work equation:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$1 \cdot \delta_{CH} = \int \theta \cdot \delta M \cdot ds$$
$$= \int \frac{M}{EI} \cdot \delta M \cdot ds$$
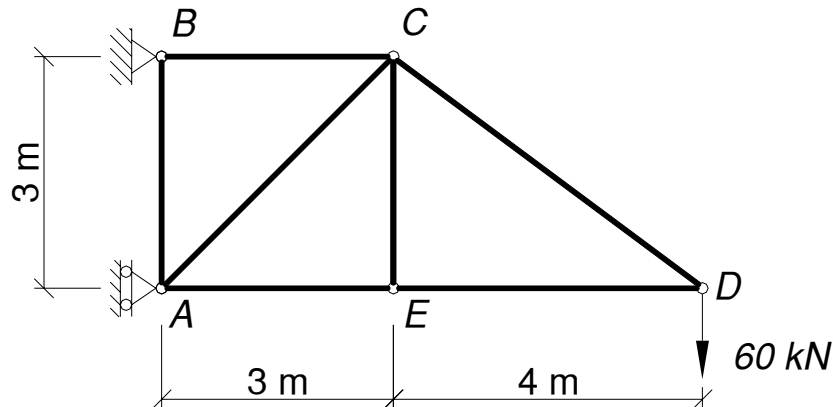
Next, using the table of volume integrals, we have:

$$\int \frac{M}{EI} \cdot \delta M \cdot ds = \frac{1}{EI} \left\{ \left[\frac{1}{3}(440)(2)\left(4\sqrt{2}\right) \right]_{AB} + \left[\frac{1}{6}(2)(160 + 2 \cdot 440)(4) \right]_{BC} \right\}$$
$$= \frac{1659.3}{EI} + \frac{1386.7}{EI}$$
$$= \frac{3046}{EI}$$

Hence:

$$1 \cdot \delta_{CH} = \frac{3046}{EI} = \frac{3046}{36 \times 10^3} \times 10^3 = 84.6 \text{ mm}$$

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4.5   Problems

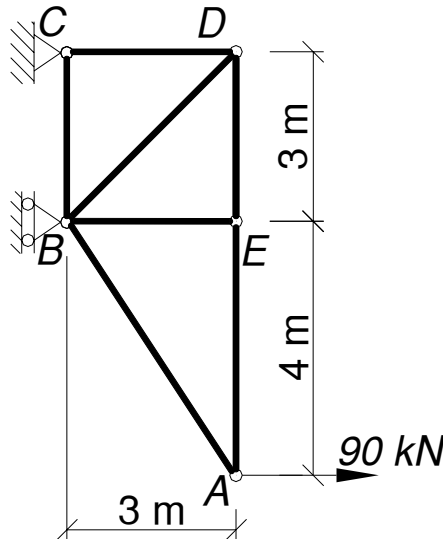1. Determine the vertical deflection of joint E of the truss shown. Take E = 200 kN/mm^2 and member areas, A = 1000 mm^2 for all members except AC where A = 1000$\sqrt{2}$ mm^2. (*Ans*. 12.2 mm).

2. Determine the vertical and horizontal deflection of joint C of the truss shown. Take E = 10 kN/mm^2 and member areas, A = 1000 mm^2 for all members except AC where A = 1000$\sqrt{2}$ mm^2 and CE where A = 2500 mm^2.

    (*Ans*. 32/7 mm horizontal, 74/7 mm vertical).

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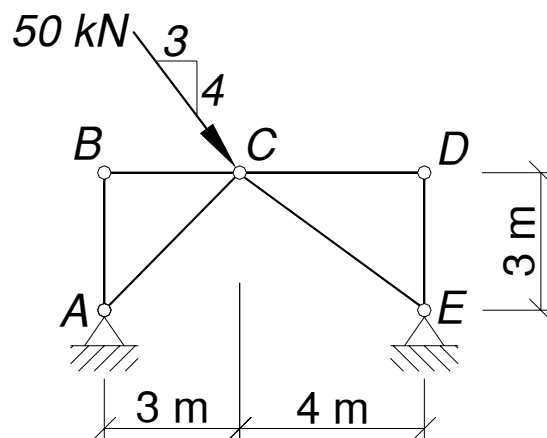
3. Determine the horizontal deflection of joint <u>A</u> and the vertical deflection of joint B of the truss shown. Take $E = 200$ kN/mm^2 and member areas, A = 1000 mm^2 for all members except *BD* where $A = 1000\sqrt{2}$ mm2 and *AB* where $A = 2500$ mm^2. (*Ans.* 15.3 mm; 0 mm)

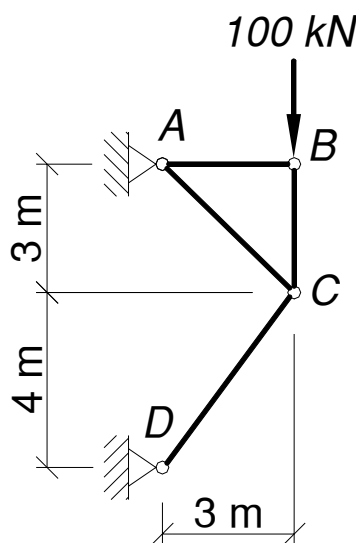4. Determine the vertical and horizontal deflection of joint *C* of the truss shown. Take $E = 200$ kN/mm^2 and member areas, $A = 150$ mm^2 for all members except *AC* where $A = 150\sqrt{2}$ mm2 and *CD* where $A = 250$ mm^2. (*Ans.* 8.78 mm; 0.2 mm)

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5. Determine the vertical deflection of joint D of the truss shown. Take $E = 200$ kN/mm² and member areas, $A = 1000$ mm² for all members except BC where $A = 1000\sqrt{2}$ mm². (*Ans.* 24 mm)
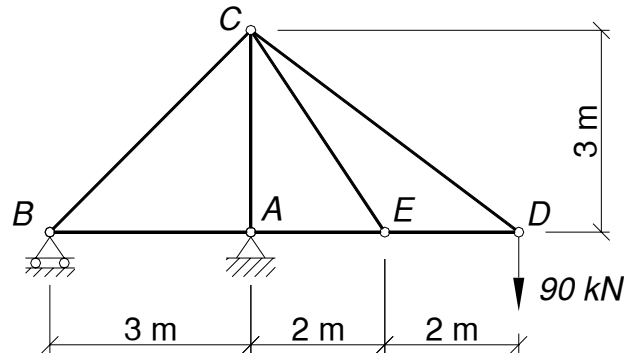
6. Verify that the deflection at the centre of a simply-supported beam under a uniformly distributed load is given by:

$$\delta_C = \frac{5wL^4}{384EI}$$

7. Show that the flexural deflection at the tip of a cantilever, when it is subjected to a point load on the end is:
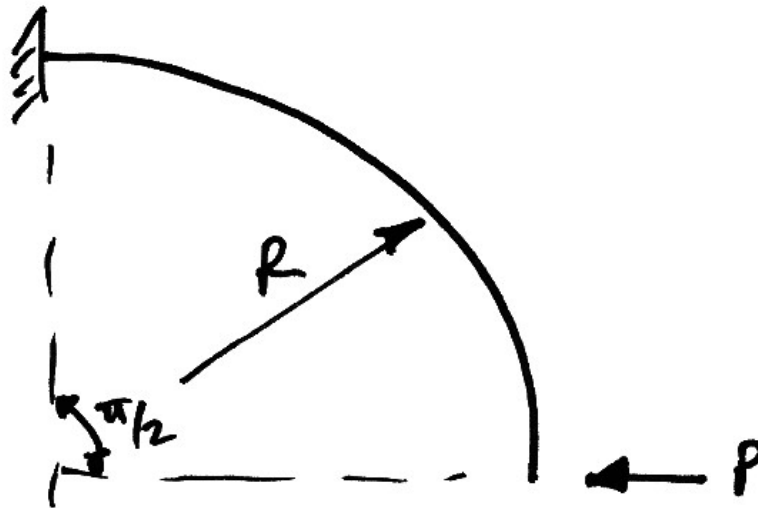
$$\delta_B = \frac{PL^3}{3EI}$$

8. Show that the rotation that occurs at the supports of a simply supported beam with a point load in the middle is:

$$\theta_C = \frac{PL^2}{16EI}$$

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9. Show that the vertical and horizontal deflections at *B* of the following frame are:

$$\delta_{By} = \frac{PR^3}{2EI}; \qquad \delta_{Bx} = \frac{PR^3}{4EI}$$

10. For the frame of Example 10, using virtual work, verify the following displacements in which the following directions are positive: vertical upwards; horizontal to the right, and; clockwise:

- Rotation at *A* is $-\dfrac{1176.3}{EI}$;

- Vertical deflection at *B* is $-\dfrac{3046}{EI}$;

- Horizontal deflection at *D* is $\dfrac{8758.7}{EI}$;

- Rotation at *D* is $\dfrac{1481.5}{EI}$.

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11. For the cantilever beam below, show that the vertical deflection of B, allowing

for both flexural and shear deformation, is $\dfrac{PL^3}{EI} + \dfrac{PL}{GA_v}$.

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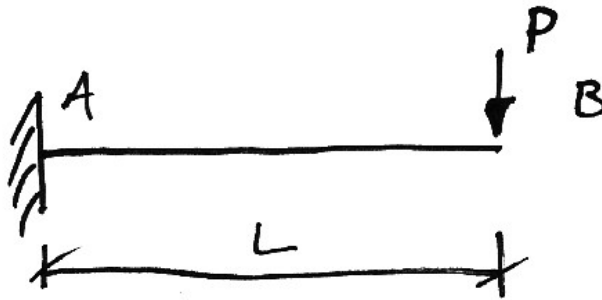
Problems 12 to 16 are considered **Genius Level**!
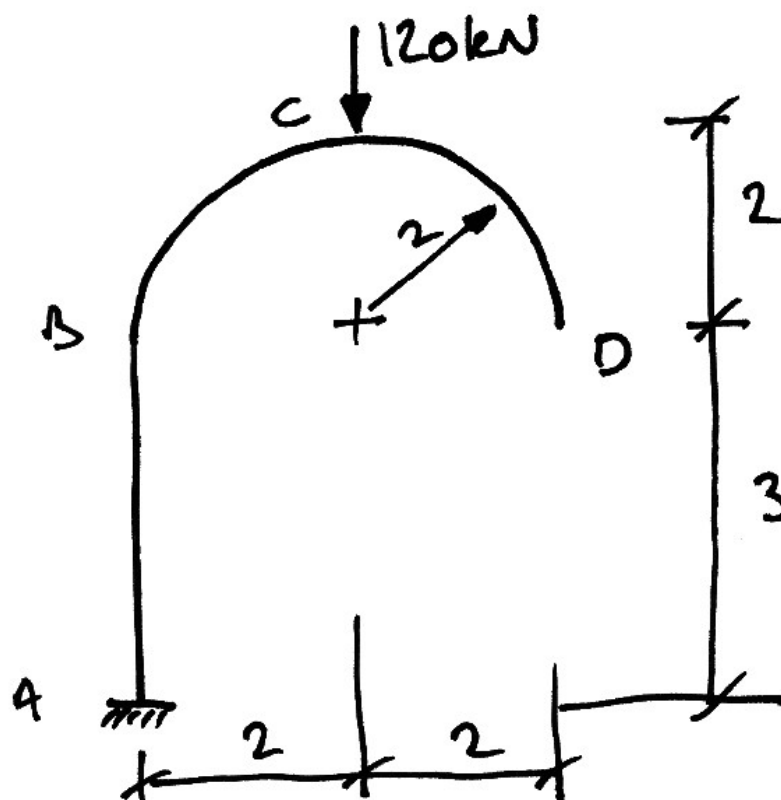
12. Show that the vertical deflection at the centre of a simply supported beam, allowing for both flexural and shear deformation is $\dfrac{5wL^4}{384EI} + \dfrac{wL^2}{8GA_v}$.

13. For the frame shown, show that the vertical deflection of point D is $\dfrac{4594}{EI}\downarrow$. Neglect axial deformation and take $EI = 120 \times 10^3$ kNm^2.

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14. For the frame shown, determine the vertical deflection of *C* and the rotation at *D* about the global *x* axis. Take the section and material properties as follows:

$A = 6600$ mm^2 $E = 205$ kN/mm^2 $G = 100$ kN/mm^2

$I = 36 \times 10^6$ mm^4 $J = 74 \times 10^6$ mm^4.
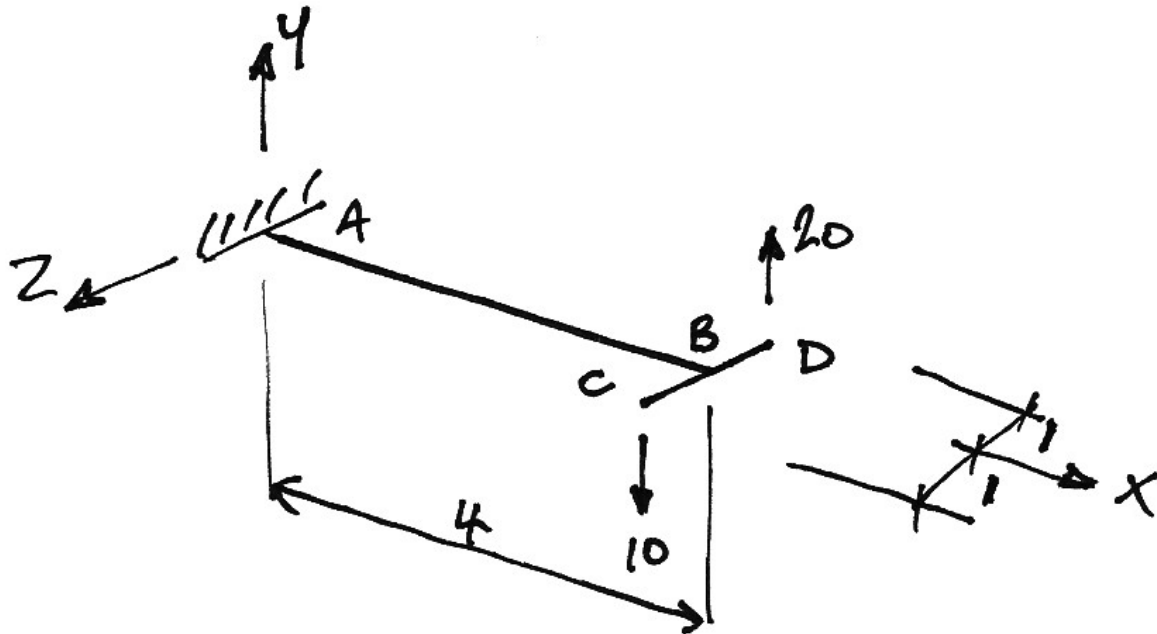
(*Ans.* 12.23 mm ↑; 0.0176 rads ACW)

15. Determine the horizontal and vertical deflections at *B* in the following frame. Neglect shear and axial effects.
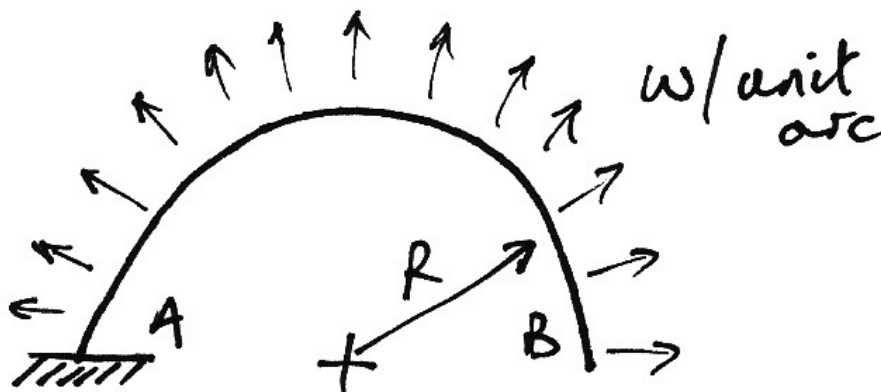
(*Ans.* $\delta_{Bx} = \dfrac{2wR^4}{EI} \rightarrow$; $\delta_{By} = \dfrac{3\pi wR^4}{2EI} \uparrow$)

16. For the frame shown, neglecting shear effects, determine the vertical deflection of points *B* and *A*.
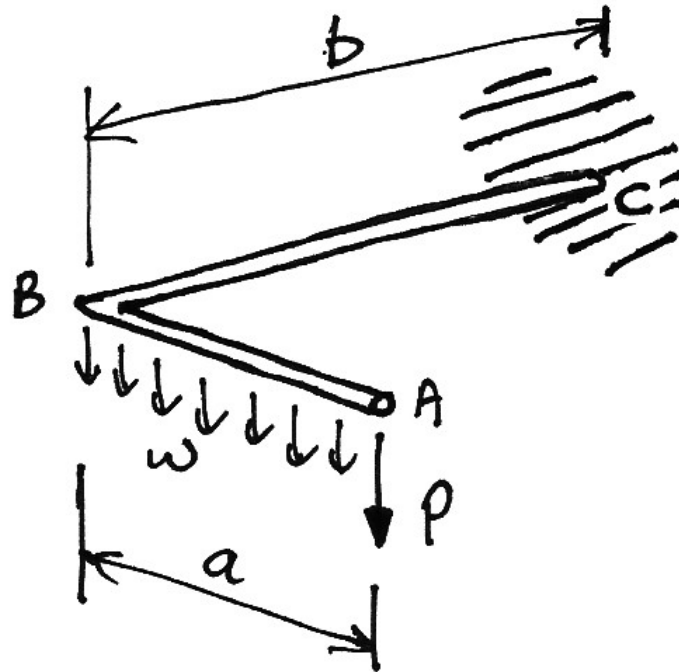
(Ans. $\delta_{By} = \dfrac{(P + wa)b^3}{3EI} \downarrow$; $\delta_{Ay} = \dfrac{1}{EI}\left[\dfrac{Pa^3}{3} + \dfrac{wa^4}{8} + \dfrac{(P + wa)b^3}{3} \right] + \left(Pa + \dfrac{wa^2}{2} \right)\dfrac{ab}{GJ} \downarrow$)

5. Virtual Work for Indeterminate Structures

5.1 General Approach

Using the concept of compatibility of displacement, any indeterminate structure can be split up into a primary and reactant structures. In the case of a one-degree indeterminate structure, we have:

Final   **=**   **Primary**   **+**   **Reactant**

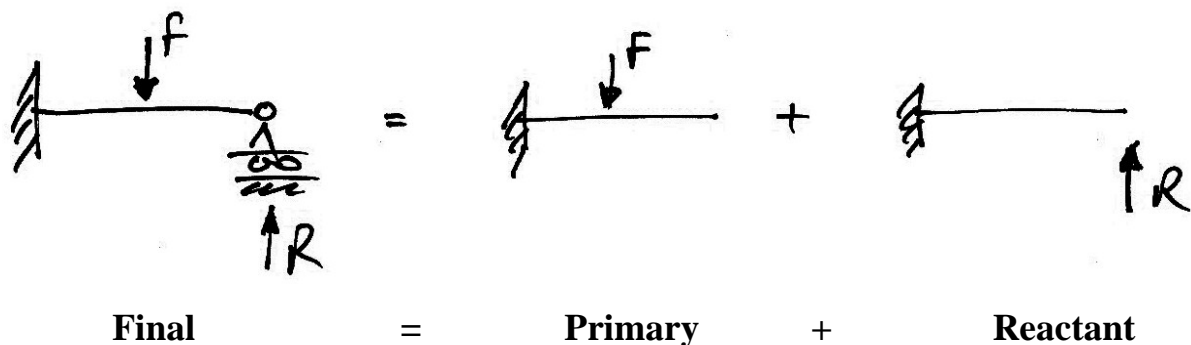At this point the primary structure can be analysed. However, we further break up the reactant structure, using linear superposition:

Reactant   **=**   **Multiplier**   **×**   **Unit Reactant**
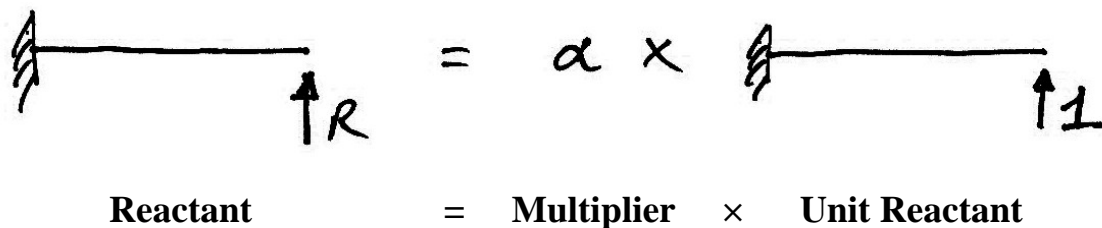
We summarize this process as:

$$M = M^0 + \alpha M^1$$

- M is the force system in the original structure (in this case moments);
- M^0 is the primary structure force system;
- M^1 is the unit reactant structure force system.

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For a truss, the procedure is the same:

Final   =   **Primary**   +   **Reactant**

Reactant   =   **Multiplier** × **Unit Reactant**
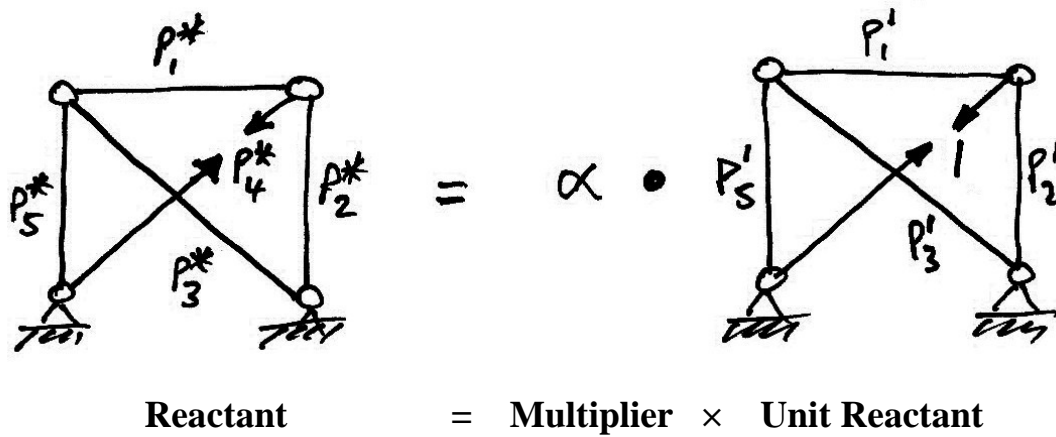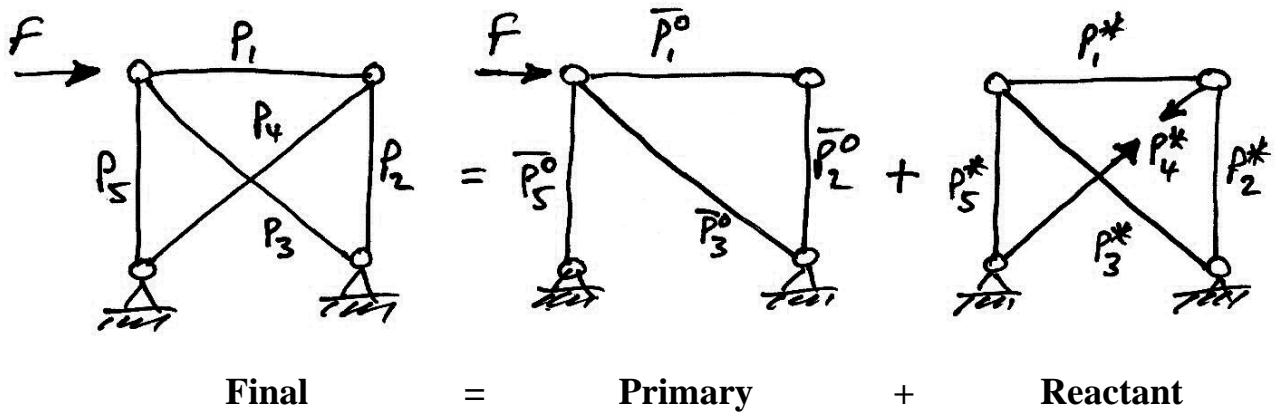
The final system forces are:

$$P = P^0 + \alpha P^1$$

The primary structure can be analysed, as can the unit reactant structure. Therefore, the only unknown is the multiplier, α.

We use virtual work to calculate the multiplier α.

5.2 Using Virtual Work to Find the Multiplier

We must identify the two sets for use:

- Displacement set: We use the actual displacements that occur in the real structure;

- Equilibrium set: We use the unit reactant structure's set of forces as the equilibrium set. We do this, as the unit reactant is always a determinate structure and has a configuration similar to that of the displacement set.

Show this process graphically:

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The virtual work equation (written for trusses) gives:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$
$$0 \cdot 1 = \sum \left(\frac{PL}{EA} \right)_i \cdot \delta P_i^1$$

There is zero external virtual work. This is because there is no external virtual force applied. Also note that the real deformations that occur in the members are in terms of P, the unknown final forces. Hence, substituting $P = P^0 + \alpha \cdot \delta P^1$ (where δ is now used to indicate virtual nature):

$$0 = \sum \left(\frac{\left(P^0 + \alpha \cdot \delta P^1 \right) L}{EA} \right)_i \cdot \delta P_i^1$$

$$= \sum \left(\frac{P^0 L}{EA} \right)_i \cdot \delta P_i^1 + \alpha \cdot \sum \left(\frac{\delta P^1 L}{EA} \right)_i \cdot \delta P_i^1$$

$$0 = \sum \frac{P^0 \cdot \delta P_i^1 \cdot L_i}{EA_i} + \alpha \cdot \sum \frac{\left(\delta P_i^1 \right)^2 L_i}{EA_i}$$

For beams and frames, the same development is:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\sum y_i \cdot \delta F_i = \sum \kappa_i \cdot \delta M_i$$
$$0 \cdot 1 = \sum \frac{M_i}{EI} \cdot \delta M_i$$

Where again there is no external displacement of the virtual force.

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Also, substitute $M = M^0 + \alpha \cdot \delta M^1$ to get:

$$0 = \sum \int_0^L \frac{\left(M^0 + \alpha \cdot \delta M^1 \right)}{EI_i} \delta M_i^1 dx$$

$$0 = \sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx + \alpha \cdot \sum \int_0^L \frac{\left(\delta M_i^1 \right)^2}{EI_i} dx$$

Thus in both bases we have a single equation with only one unknown, α. We can establish values for the other two terms and then solve for α and the structure as a whole.
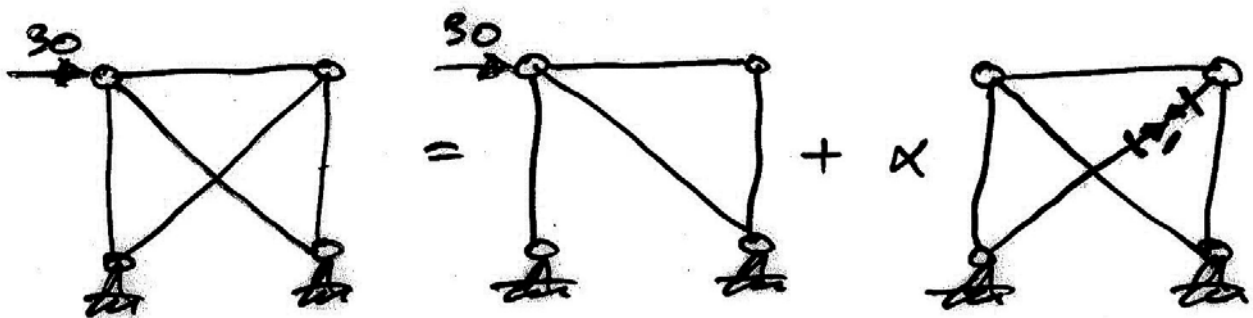
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5.3 Indeterminate Trusses

Example 11

Calculate the forces in the truss shown and find the horizontal deflection at *C*. Take *EA* to be 10×10^4 kN for all members.
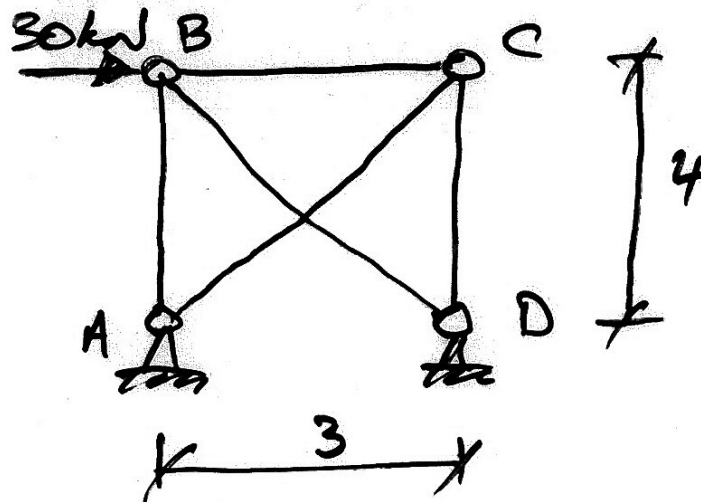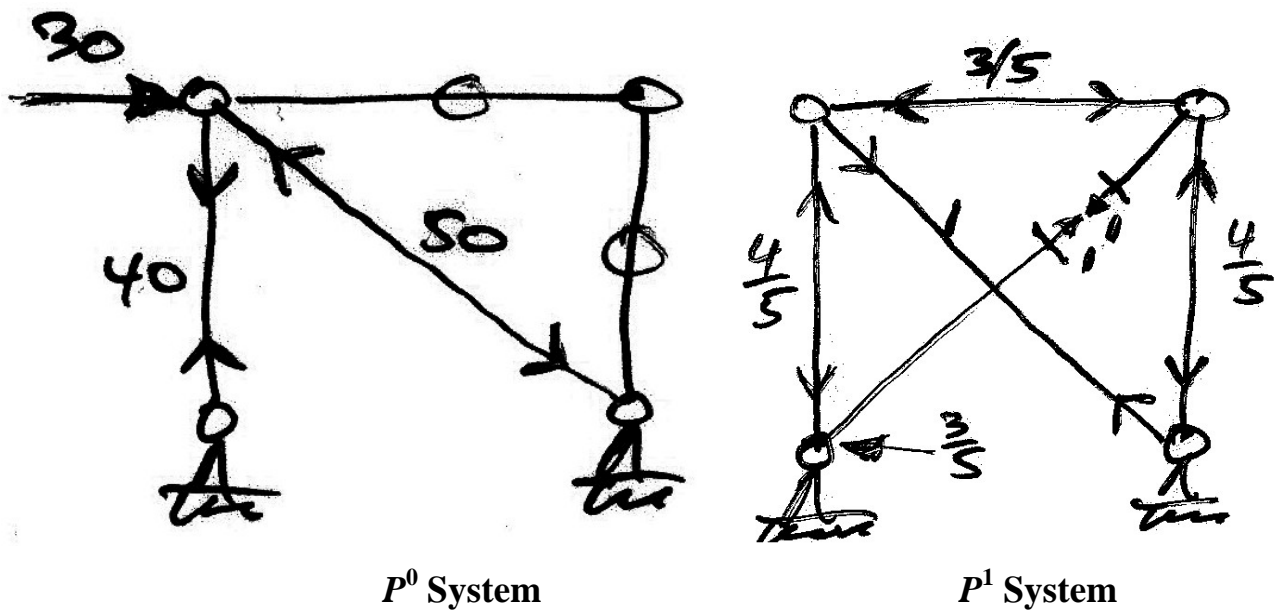

Solve for the Truss Member Forces

Choose member *AC* as the redundant:

Next analyse for the P^0 and P^1 force systems.

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P⁰ System   **P¹ System**

Using a table:

Member	L	P^0	δP^1	$P^0 \cdot \delta P^1 \cdot L$	$\left(\delta P^1\right)^2 \cdot L$
	(mm)	(kN)	(kN)	$\times 10^4$	$\times 10^4$
AB	4000	+40	- 4/5	-12.8	0.265
BC	3000	0	- 3/5	0	0.108
CD	4000	0	- 4/5	0	0.256
AC	5000	0	1	0	0.5
BD	5000	-50	1	-25	0.5
			$\sum =$	-37.8	1.62

Hence:

$$0 = \sum \frac{P^0 \cdot \delta P_i^1 \cdot L_i}{EA_i} + \alpha \cdot \sum \frac{\left(\delta P_i^1\right)^2 L_i}{EA_i}$$

$$= \frac{-37.8 \times 10^4}{EA} + \alpha \cdot \frac{1.62 \times 10^4}{EA}$$

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And so

$$\alpha = -\frac{-37.8}{1.62} = 23.33$$

The remaining forces are obtained from the compatibility equation:

| Member	P^0	δP^1	$P = P^0 + \alpha \cdot \delta P^1$
	(kN)	(kN)	(kN)
AB	+40	- 4/5	21.36
BC	0	- 3/5	-14
CD	0	- 4/5	-18.67
AC	0	1	23.33
BD	-50	1	-26.67

Note that the redundant always has a force the same as the multiplier.

Calculate the Horizontal Deflection at *C*

To calculate the horizontal deflection at *C*, using virtual work, the two relevant sets are:

- Compatibility set: the actual deflection at *C* and the real deformations that occur in the actual structure;
- Equilibrium set: a horizontal unit virtual force applied at *C* to a portion of the actual structure, yet ensuring equilibrium.

We do not have to apply the virtual force to the full structure. Remembering that the only requirement on the virtual force system is that it is in equilibrium; choose the force systems as follows:

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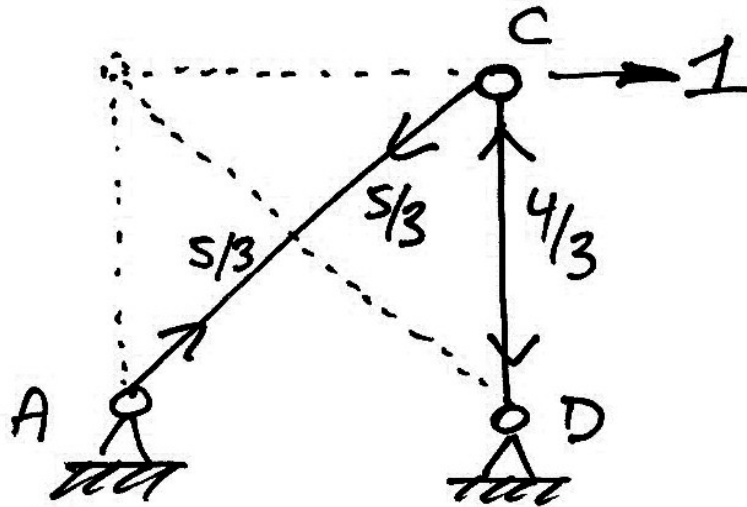
Thus we have:

$$\delta W = 0$$

$$\delta W_E = \delta W_I$$

$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$

$$y_{CH} \cdot 1 = \sum \left(\frac{PL}{EA} \right)_i \cdot \delta P_i$$

$$= \left(\frac{23.33 \times 5000}{10 \times 10^4} \right) \cdot \frac{5}{3} + \left(\frac{-18.67 \times 4000}{10 \times 10^4} \right) \cdot \frac{-4}{3}$$

$$y_{CH} = 2.94 \text{ mm}$$

Because we have chosen only two members for our virtual force system, only these members do work and the calculation is greatly simplified.

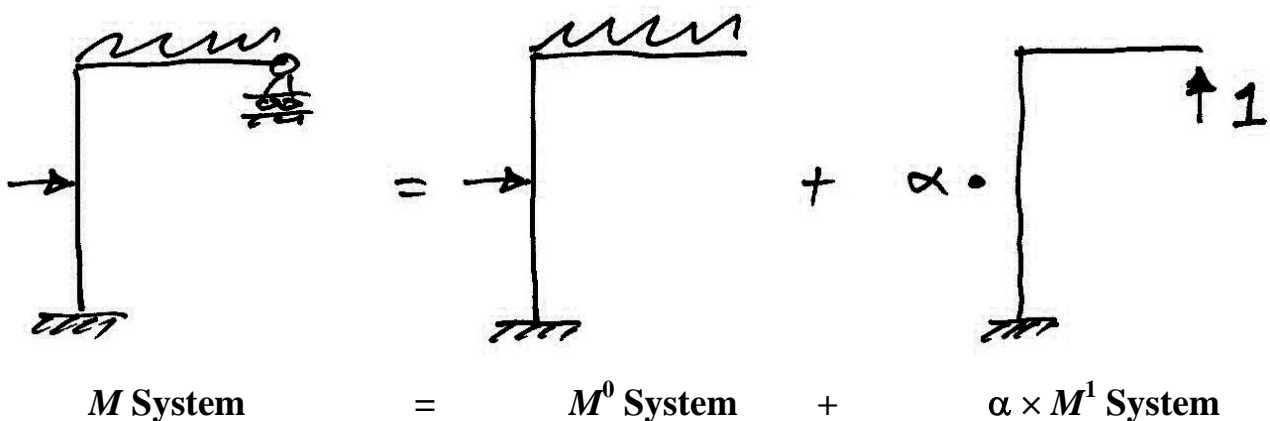*Dr. C. Caprani*
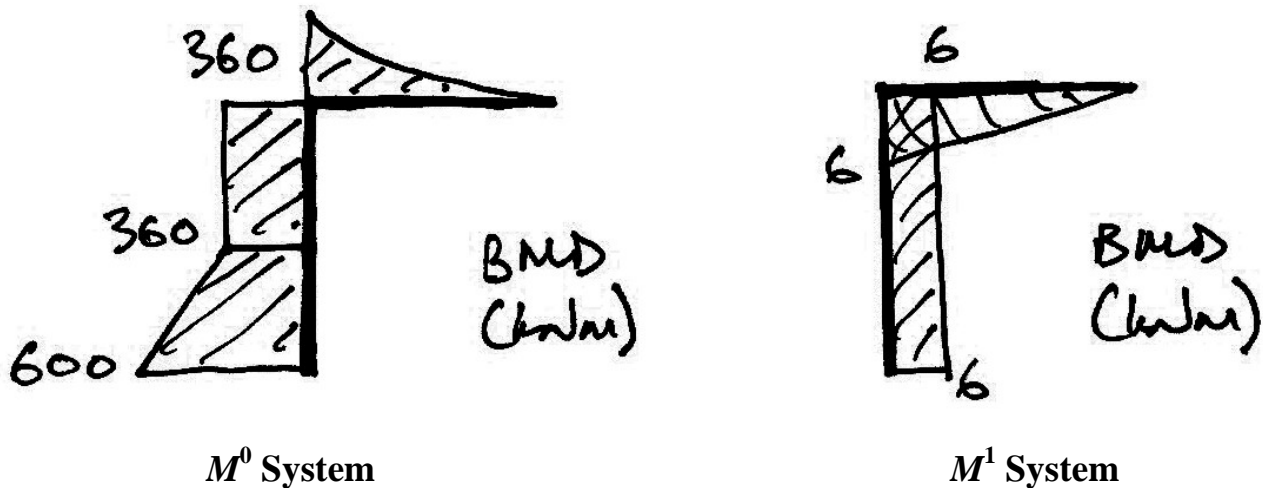
5.4   Indeterminate Frames

Example 12

For the frame shown, calculate the reactions and draw the bending moment diagram. Determine the horizontal deflection at joint C. Take $EI = 40 \times 10^3$ kNm^2, constant for all members.
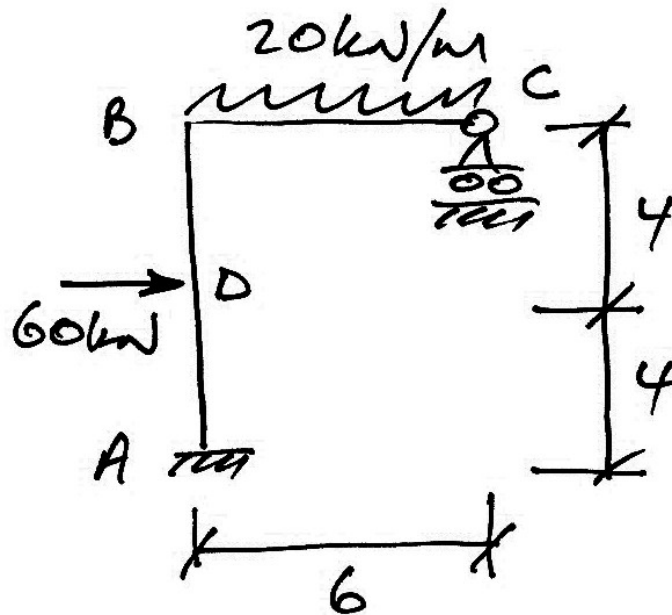
Break the frame up into its reactant and primary structures:

M System   =   **M^0 System**   +   **$\alpha \times M^1$ System**

Establish the M^0 and M^1 force systems:

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M^0 System   **M^1 System**

Apply the virtual work equation:

$$0 = \sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx + \alpha \cdot \sum \int_0^L \frac{\left(\delta M_i^1\right)^2}{EI_i} dx$$

We will be using the table of volume integrals to quicken calculations. Therefore we can only consider lengths of members for which the correct shape of bending moment diagram is available. Also, we must choose sign convention: we consider tension on the outside of the frame to be positive.

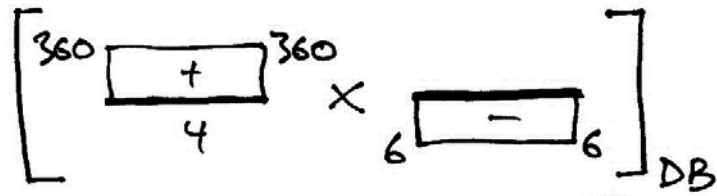As each term has several components we consider them separately:

Term 1 - $\sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx$ **:**

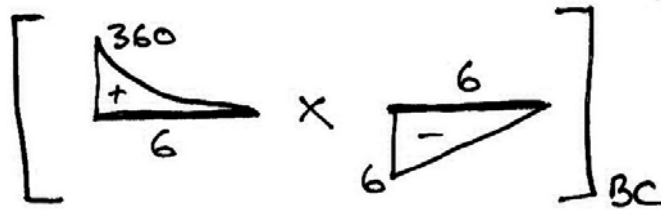- AD: we have graphically, and from the volume integral table:
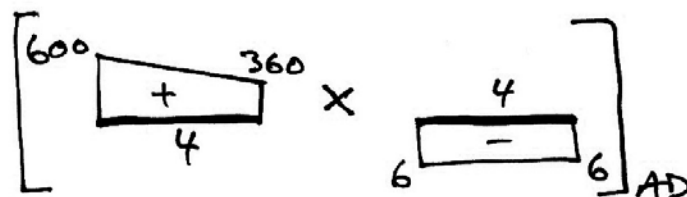
$$\frac{1}{2} j(k_1 + k_2) l = \frac{1}{2}(-6)(600 + 360)4 = -11520$$

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- DB: Similarly:

$$jkl = (-6)(360)4 = -8640$$

- BC:

$$\frac{1}{4} jkl = \frac{1}{4}(-6)(360)6 = -3240$$

Hence:

$$\sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx = \frac{-23400}{EI}$$

Term 2:

$$\sum \int_0^L \frac{\left(\delta M_i^1\right)^2}{EI_i} dx = \frac{1}{EI}\left\{\left[jkl\right]_{AB} + \left[\frac{1}{3} jkl\right]_{BC}\right\}$$

$$= \frac{1}{EI}\left\{\left[(-6)(-6)8\right]_{AB} + \left[\frac{1}{3}(-6)(-6)6\right]_{BC}\right\}$$

$$= \frac{360}{EI}$$

Note that Term 2 is always easier to calculate as it is only ever made up of straight line bending moment diagrams.

Thus we have:

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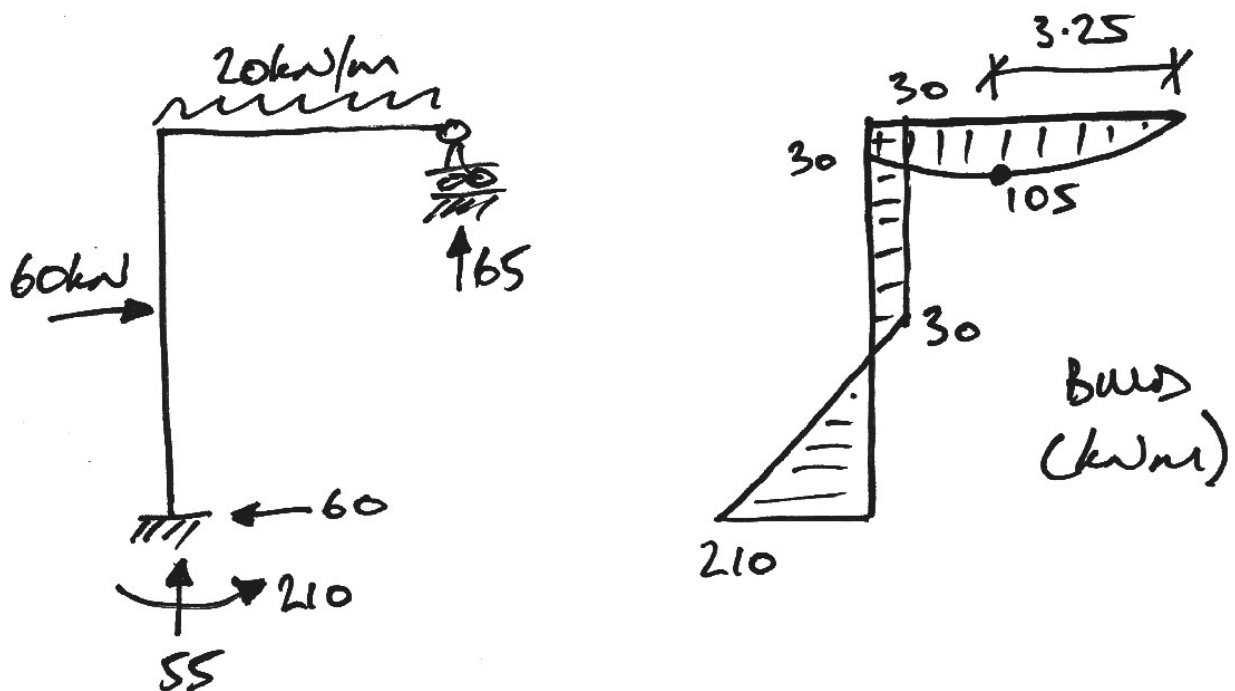
$$0 = \sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx + \alpha \cdot \sum \int_0^L \frac{\left(\delta M_i^1\right)^2}{EI_i} dx$$

$$= \frac{-23400}{EI} + \alpha \cdot \frac{360}{EI}$$

And so

$$\alpha = \frac{23400}{360} = 65.0$$

Thus the vertical reaction at C is 65.0 kN upwards. With this information we can solve for the moments using $M = M^0 + \alpha M^1$ (or by just using statics) and the remaining reactions to get:

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To calculate the horizontal deflection at *C* using virtual work, the two relevant sets are:

- Compatibility set: the actual deflection at *C* and the real deformations (rotations) that occur in the actual structure;

- Equilibrium set: a horizontal unit virtual force applied at *C* to a determinate portion of the actual structure.

Choose the following force system as it is easily solved:
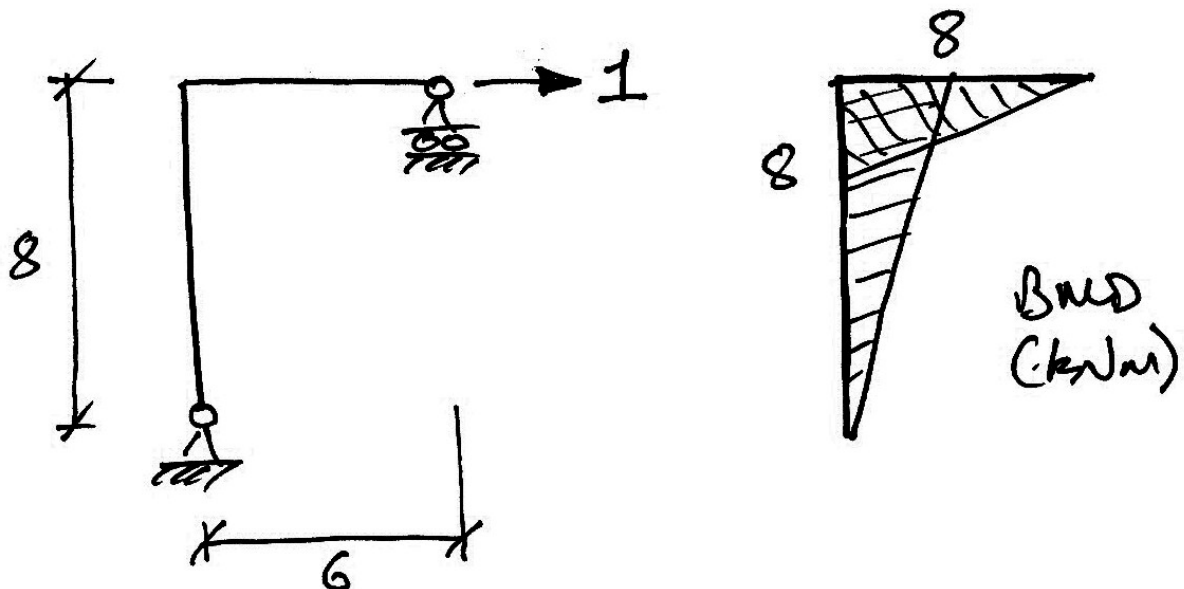
Thus we have:

$$\delta W = 0$$

$$\delta W_E = \delta W_I$$

$$\sum y_i \cdot \delta F_i = \sum \theta_i \cdot \delta M_i$$

$$y_{CH} \cdot 1 = \sum \int_0^L \left[\frac{M_x}{EI} \right] \cdot \delta M_x \, dx$$

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The real bending moment diagram, M, is awkward to use with the integral table. Remembering that $M = M^0 + \alpha M^1$ simplifies the calculation by using:

To give:

And so using the table formulae, we have:

$$
\left[\frac{1}{2}(-4)(-6\cdot\alpha)4 + \frac{1}{6}(2\cdot360+600)(-4)4 \right]_{AD} +
$$

$$
\left[\frac{1}{2}(-4-8)(-6\cdot\alpha)4 + \frac{1}{2}(-4-8)(360)4 \right]_{DB} +
$$

$$
\left[\frac{1}{3}(-8)(-6\cdot\alpha)6 + \frac{1}{4}(-8)(360)6 \right]_{BC}
$$

Which gives us:

$$
\sum \int_0^L \left[\frac{M_x}{EI} \right] \cdot \delta M_x \, dx = \frac{1}{EI} \left[288\alpha + (-16480) \right]
$$

$$
= \frac{2240}{EI}
$$

$$
= 0.056
$$

$$
= 56 \text{ mm}
$$

The answer is positive, indicating that the structure moves to the right at C: the same direction in which the unit virtual force was applied.

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5.5 Continuous Beams

Basis

In this section we will introduce structures that are more than 1 degree statically indeterminate. We do so to show that virtual work is easily extensible to multiply-indeterminate structures, and also to give a method for such beams that is easily worked out, and put into a spreadsheet.

Consider the example 3-span beam. It is 2 degrees indeterminate, and so we introduce 2 hinges (i.e. moment releases) at the support locations and unit reactant moments in their place, as shown:

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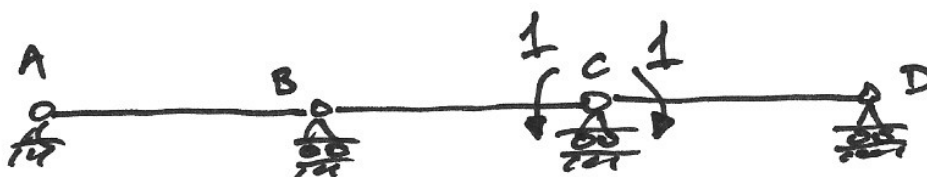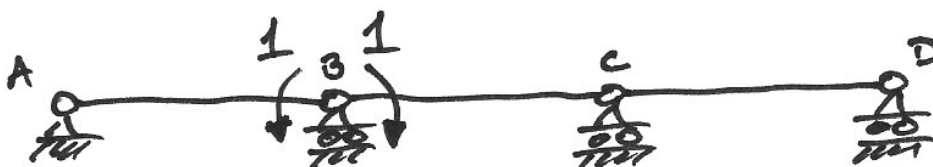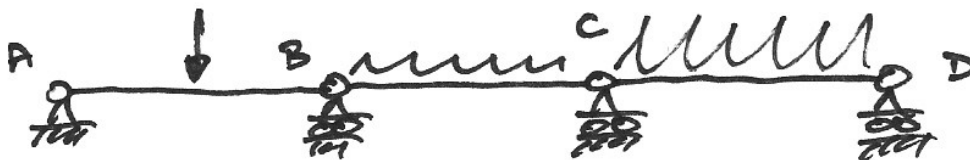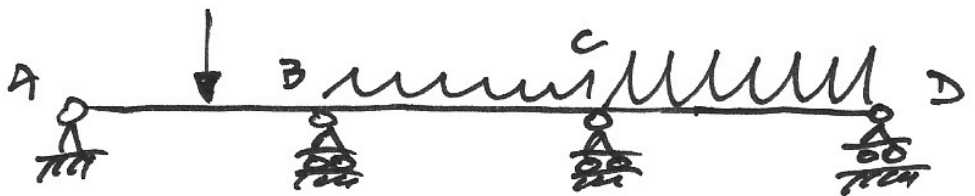
Alongside these systems, we have their bending moment diagrams:
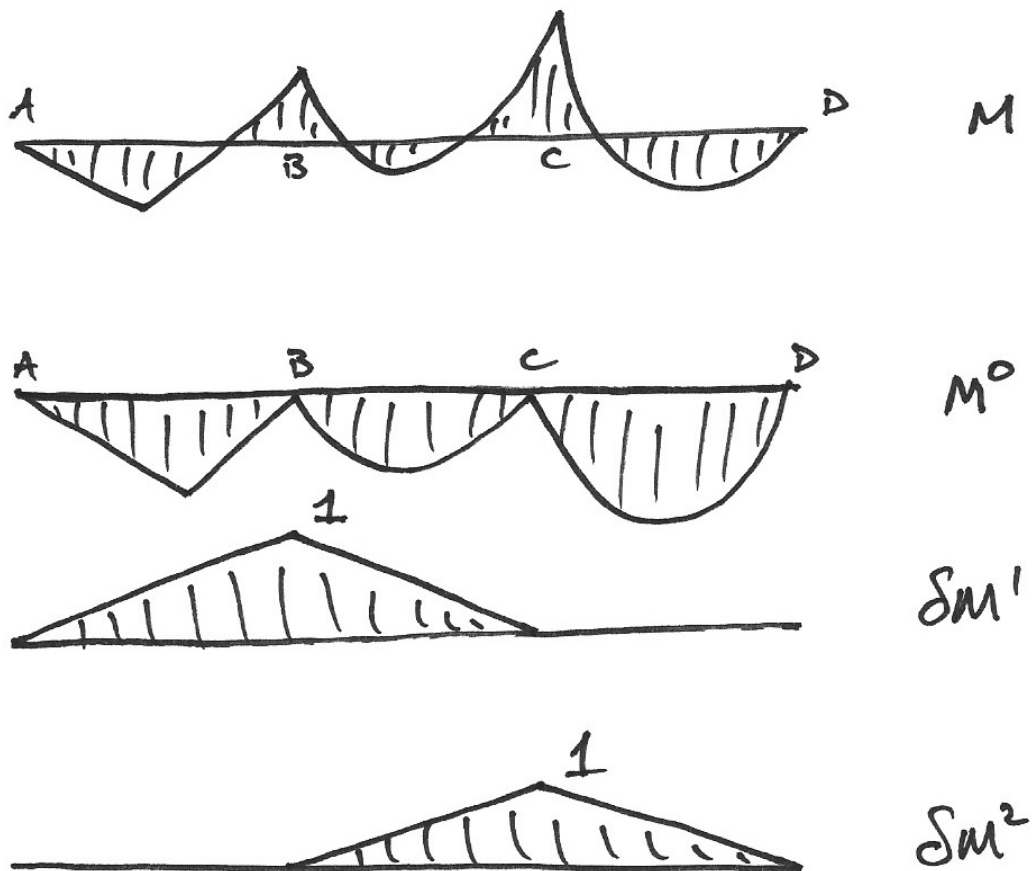
Using the idea of the multiplier and superposition again, we can see that:

$$M = M^0 + \alpha_1 \cdot \delta M^1 + \alpha_2 \cdot \delta M^2$$

The virtual work equation is:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$0 \cdot 1 = \sum \int \theta \cdot \delta M$$

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There is no external virtual work done since the unit moments are applied internally. Since we have two virtual force systems in equilibrium and one real compatible system, we have two equations:

$$0 = \sum \int \frac{M}{EI} \cdot \delta M^1 \quad \text{and} \quad 0 = \sum \int \frac{M}{EI} \cdot \delta M^2$$

For the first equation, expanding the expression for the real moment system, M:

$$\sum \int \frac{\left(M^0 + \alpha_1 \cdot \delta M^1 + \alpha_2 \cdot \delta M^2 \right)}{EI} \cdot \delta M^1 = 0$$

$$\int \frac{M^0 \cdot \delta M^1}{EI} + \alpha_1 \int \frac{\delta M^1 \cdot \delta M^1}{EI} + \alpha_2 \int \frac{\delta M^2 \cdot \delta M^1}{EI} = 0$$

In which we've dropped the summation over all members – it being understood that we sum for all members. Similarly for the second virtual moments, we have:

$$\int \frac{M^0 \cdot \delta M^2}{EI} + \alpha_1 \int \frac{\delta M^1 \cdot \delta M^2}{EI} + \alpha_2 \int \frac{\delta M^2 \cdot \delta M^2}{EI} = 0$$

Thus we have two equations and so we can solve for α_1 and α_2. Usually we write this as a matrix equation:

$$\begin{Bmatrix} \int \dfrac{M^0 \cdot \delta M^1}{EI} \\[2mm] \int \dfrac{M^0 \cdot \delta M^2}{EI} \end{Bmatrix} + \begin{bmatrix} \int \dfrac{\delta M^1 \cdot \delta M^1}{EI} & \int \dfrac{\delta M^2 \cdot \delta M^1}{EI} \\[2mm] \int \dfrac{\delta M^1 \cdot \delta M^2}{EI} & \int \dfrac{\delta M^2 \cdot \delta M^2}{EI} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Each of the integral terms is easily found using the integral tables, and the equation solved.

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Similarly the (simpler) equation for a 2-span beam is:

$$\int \frac{M^0 \cdot \delta M^1}{EI} + \alpha_1 \int \frac{\delta M^1 \cdot \delta M^1}{EI} = 0$$

And the equation for a 4-span beam is:

$$\left\{ \begin{array}{c} \int \dfrac{M^0 \cdot \delta M^1}{EI} \\[2mm] \int \dfrac{M^0 \cdot \delta M^2}{EI} \\[2mm] \int \dfrac{M^0 \cdot \delta M^3}{EI} \end{array} \right\} + \left[\begin{array}{ccc} \int \dfrac{\delta M^1 \cdot \delta M^1}{EI} & \int \dfrac{\delta M^2 \cdot \delta M^1}{EI} & 0 \\[2mm] \int \dfrac{\delta M^1 \cdot \delta M^2}{EI} & \int \dfrac{\delta M^2 \cdot \delta M^2}{EI} & \int \dfrac{\delta M^3 \cdot \delta M^2}{EI} \\[2mm] 0 & \int \dfrac{\delta M^2 \cdot \delta M^3}{EI} & \int \dfrac{\delta M^3 \cdot \delta M^3}{EI} \end{array} \right] \left\{ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\}$$

Sketch out why there are no terms involving $\delta M^3 \cdot \delta M^1$:

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Example 13

Using virtual work, find the bending moment diagram for the following beam:
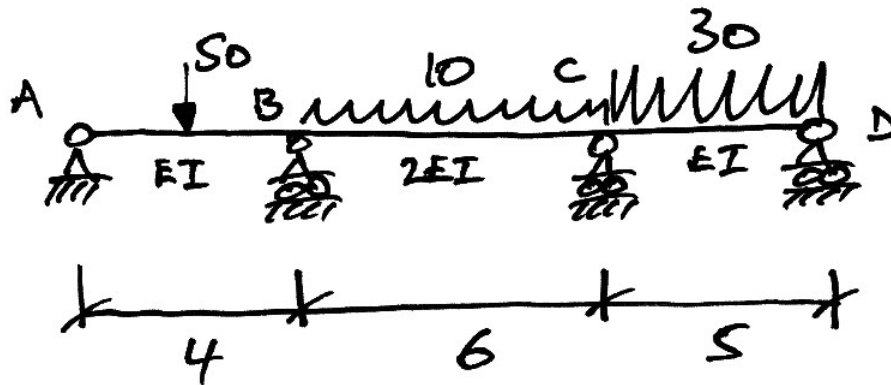
Proceeding as described above, we introduce releases (hinges) at the support points, apply the unit virtual moments, and find the corresponding bending moment diagrams:
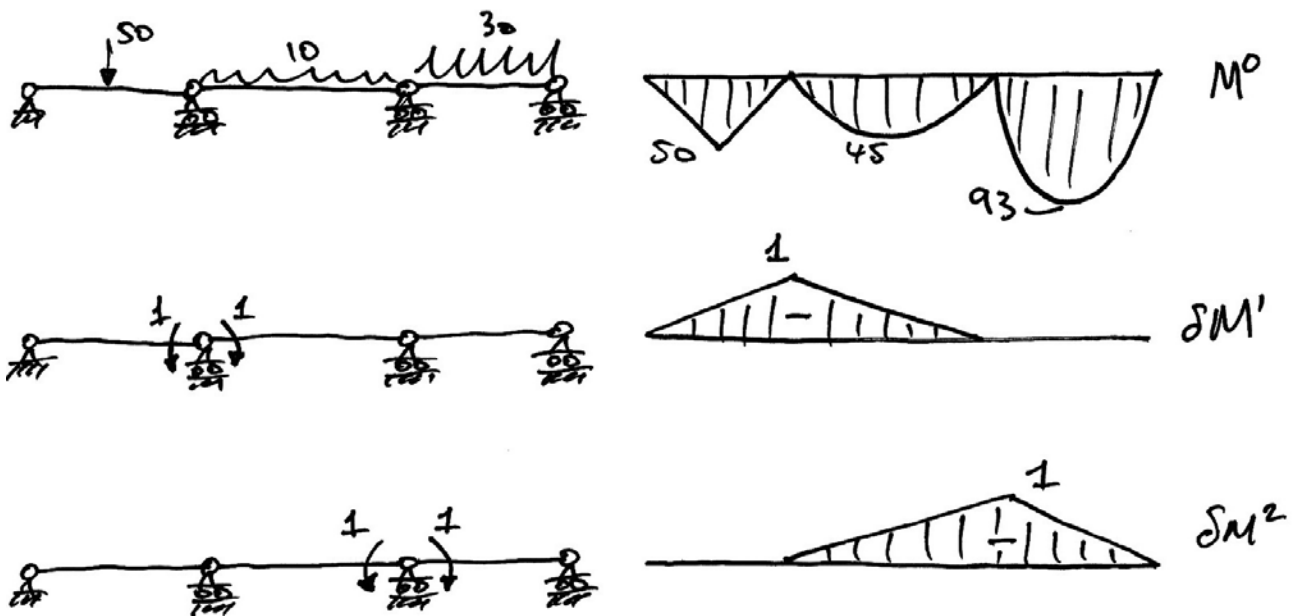
Next we need to evaluate each term in the matrix virtual work equation. We'll take the two 'hard' ones first:

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$$\int \frac{M^0 \cdot \delta M^1}{EI} = \frac{1}{EI}\left[\frac{1}{6}(50)(-1)(4+2)\right]_{AB} + \frac{1}{2EI}\left[\frac{1}{3}(45)(-1)(6)\right]_{BC}$$

$$= \frac{1}{EI}(-50-45) = -\frac{95}{EI}$$

Note that since $\delta M^1 = 0$ for span *CD*, there is no term for it above. Similarly, for the following evaluation, there will be no term for span *AB*:

$$\int \frac{M^0 \cdot \delta M^2}{EI} = \frac{1}{2EI}\left[\frac{1}{3}(45)(-1)(6)\right]_{BC} + \frac{1}{EI}\left[\frac{1}{3}(93.75)(-1)(5)\right]_{CD}$$

$$= \frac{1}{EI}(-45-156.25) = -\frac{201.25}{EI}$$

The following integrals are more straightforward since they are all triangles:

$$\int \frac{\delta M^1 \cdot \delta M^1}{EI} = \frac{1}{EI}\left[\frac{1}{3}(-1)(-1)(4)\right]_{AB} + \frac{1}{2EI}\left[\frac{1}{3}(-1)(-1)(6)\right]_{BC} = \frac{2.333}{EI}$$

$$\int \frac{\delta M^2 \cdot \delta M^1}{EI} = \frac{1}{2EI}\left[\frac{1}{6}(-1)(-1)(6)\right]_{BC} = \frac{0.5}{EI}$$

$$\int \frac{\delta M^1 \cdot \delta M^2}{EI} = \frac{0.5}{EI}, \text{ since it is equal to } \int \frac{\delta M^2 \cdot \delta M^1}{EI} \text{ by the commutative}$$

property of multiplication.

$$\int \frac{\delta M^2 \cdot \delta M^2}{EI} = \frac{1}{2EI}\left[\frac{1}{3}(-1)(-1)(6)\right]_{BC} + \frac{1}{EI}\left[\frac{1}{3}(-1)(-1)(5)\right]_{CD} = \frac{2.667}{EI}$$

With all the terms evaluated, enter them into the matrix equation:

$$\left\{ \begin{matrix} \int \dfrac{M^0 \cdot \delta M^1}{EI} \\[2mm] \int \dfrac{M^0 \cdot \delta M^2}{EI} \end{matrix} \right\} + \left[\begin{matrix} \int \dfrac{\delta M^1 \cdot \delta M^1}{EI} & \int \dfrac{\delta M^2 \cdot \delta M^1}{EI} \\[2mm] \int \dfrac{\delta M^1 \cdot \delta M^2}{EI} & \int \dfrac{\delta M^2 \cdot \delta M^2}{EI} \end{matrix} \right] \left\{ \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix} \right\} = \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\}$$

To give:

$$\frac{1}{EI}\left\{ \begin{matrix} -95 \\ -201.25 \end{matrix} \right\} + \frac{1}{EI}\left[\begin{matrix} 2.333 & 0.5 \\ 0.5 & 2.667 \end{matrix} \right]\left\{ \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix} \right\} = \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\}$$

And solve, as follows:

$$\left[\begin{matrix} 2.333 & 0.5 \\ 0.5 & 2.667 \end{matrix} \right]\left\{ \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix} \right\} = \left\{ \begin{matrix} 95 \\ 201.25 \end{matrix} \right\}$$

$$\left\{ \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix} \right\} = \frac{1}{(2.333 \cdot 2.667 - 0.5 \cdot 0.5)}\left[\begin{matrix} 2.667 & -0.5 \\ -0.5 & 2.333 \end{matrix} \right]\left\{ \begin{matrix} 95 \\ 201.25 \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} 25.57 \\ 70.67 \end{matrix} \right\}$$
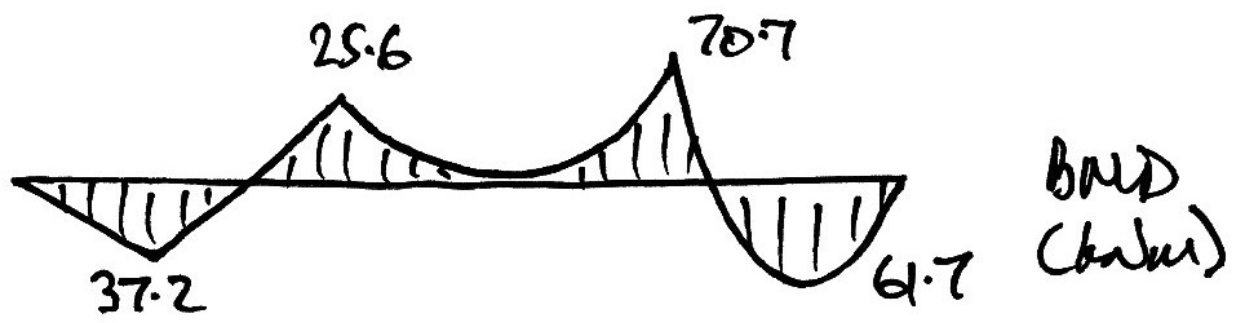
Now using our superposition equation for moments, $M = M^0 + \alpha_1 \cdot \delta M^1 + \alpha_2 \cdot \delta M^2$, we can show that the multipliers are just the hogging support moments:

$$M_B = 0 + 25.57 \cdot 1 + 70.67 \cdot 0 = 25.57 \text{ kNm}$$
$$M_C = 0 + 25.57 \cdot 0 + 70.67 \cdot 1 = 70.67 \text{ kNm}$$

From these we get the final BMD:

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Spreadsheet Solution

A simple spreadsheet for a 3-span beam with centre-span point load and UDL capabilities, showing Example 13, is:

		Span 1	Span 2	Span 3
Length	m	4	6	5
EI x		1	2	1
PL	kN	50	0	0
UDL	kN/m	0	10	30
BMDs	PL	50	0	0
	UDL	0	45	93.75

VW Equations

$$\begin{bmatrix} -95.000 \\ -201.250 \end{bmatrix} + \begin{bmatrix} 2.333 & 0.500 \\ 0.500 & 2.667 \end{bmatrix} * \begin{bmatrix} alpha1 \\ alpha2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2.333 & 0.500 \\ 0.500 & 2.667 \end{bmatrix} * \begin{bmatrix} alpha1 \\ alpha2 \end{bmatrix} = \begin{bmatrix} 95.000 \\ 201.250 \end{bmatrix}$$

$$\begin{bmatrix} alpha1 \\ alpha2 \end{bmatrix} = \begin{bmatrix} 0.446512 & -0.083721 \\ -0.083721 & 0.390698 \end{bmatrix} * \begin{bmatrix} 95.000 \\ 201.250 \end{bmatrix}$$

$$\begin{bmatrix} alpha1 \\ alpha2 \end{bmatrix} = \begin{bmatrix} 25.57 \\ 70.67 \end{bmatrix}$$

M0 x M1	Span 1	Span 2	Span 3	Totals
PL	-50.000	0.000	0.000	
UDL	0.000	-90.000	0.000	
Total	-50.000	-45.000	0.000	-95.000

M1 x M1	Span 1	Span 2	Span 3	
Total	1.333	1.000	0.000	2.333

M2 x M1	Span 1	Span 2	Span 3	
Total	0.000	0.500	0.000	0.500

M0 x M2	Span 1	Span 2	Span 3	
PL	0.000	0.000	0.000	
UDL	0.000	-90.000	-156.250	
Total	0.000	-45.000	-156.250	-201.250

M1 x M2	Span 1	Span 2	Span 3	
Total	0.000	0.500	0.000	0.500

M2 x M2	Span 1	Span 2	Span 3	
Total	0.000	1.000	1.667	2.667

Mid-Span and Support Moments

Mab	37.2	kNm
Mb	-25.6	kNm
Mbc	-3.1	kNm
Mc	-70.7	kNm
Mcd	58.4	kNm

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5.6 Problems

1. For the truss shown, calculate the vertical deflection at *C* when member *EB* is not present and when it is present. Members *AB*, *DC* and *DE* have *EA* = 60 × 10³ kN; members *EB*, *BC* and *AD* have *EA* = 100 × 10³ kN, and; member *BD* has *EA* = 80 × 10³ kN. (*Ans*. 15 mm and 11.28 mm)
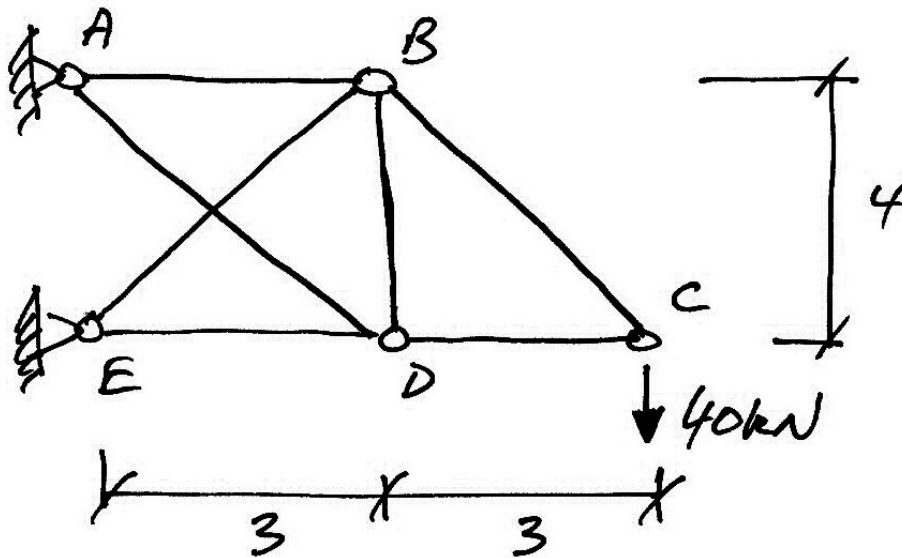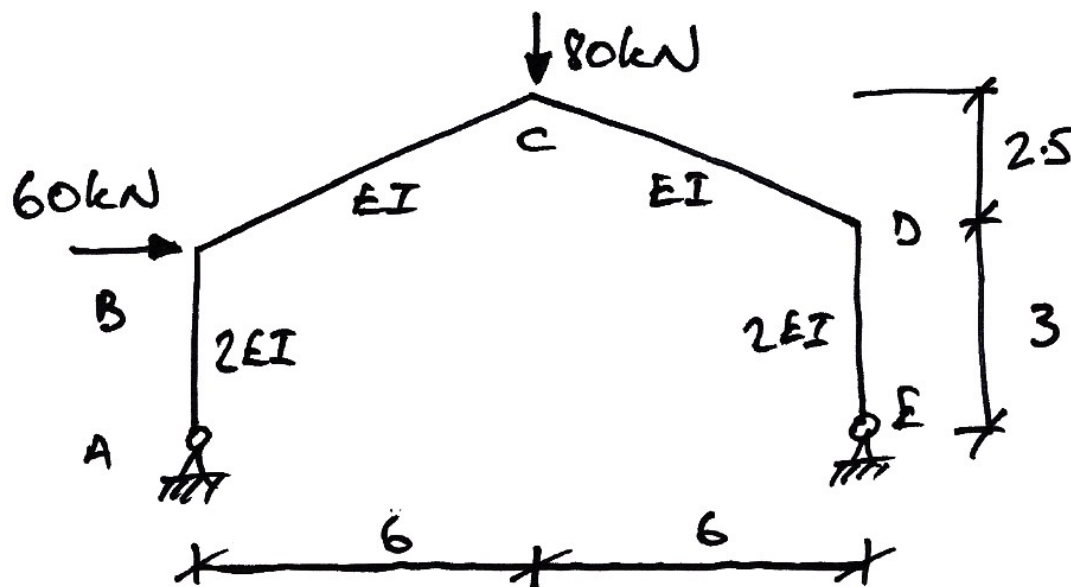

2. **Autumn 2007** For the truss shown, determine the force in each member and determine the horizontal deflection of joint B. Take $EA = 200 \times 10^3$ kN for all members.   (*Ans*. Choosing *BD*: $\alpha = 50\sqrt{2}$; $\sqrt{2}$ mm)
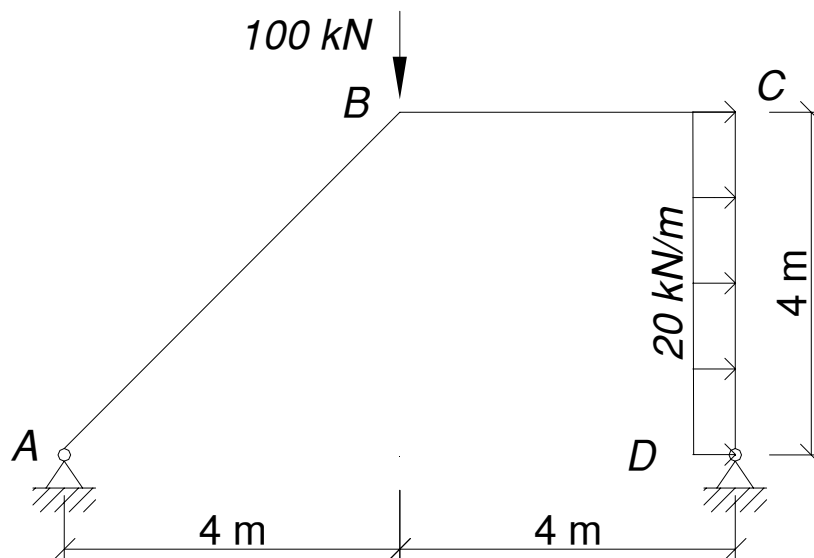

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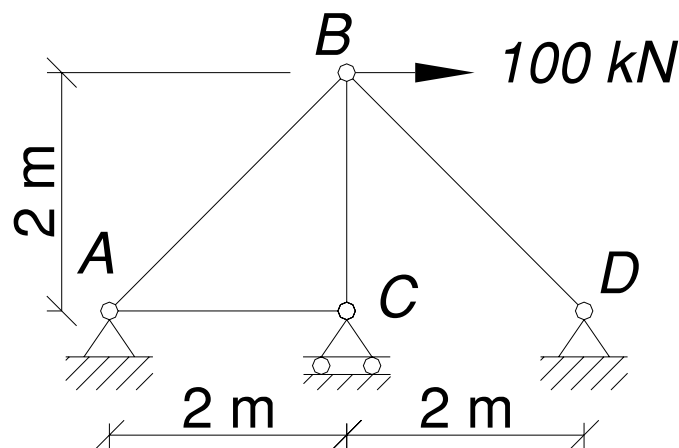
3. For the frame shown, calculate the bending moment diagram, and verify the following deflections: $\delta_{Bx} = \dfrac{856.2}{EI} \rightarrow$ and $\delta_{Cy} = \dfrac{327.6}{EI} \downarrow$. (*Ans.* $M_B = 30.1$ kNm)


4. **Summer 2007, Part (b)**: For the frame shown, draw the bending moment diagram and determine the horizontal deflection of joint C. Neglect axial effects in the members and take $EI = 36 \times 10^3$ kNm^2. (*Ans.* $H_D = 75.8$ kN; 19.0 mm)

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5. For the beam of Example 13, show that the deflections at the centre of each span, taking downwards as positive, are:

- Span AB: $\dfrac{41.1}{EI}$;

- Span BC: $-\dfrac{23.9}{EI}$;

- Span CD: $\dfrac{133.72}{EI}$.

6. Using virtual work, analyse the following prismatic beam to show that the support moments are $M_B = 51.3 \text{ kNm}$ and $M_C = 31.1 \text{ kNm}$, and draw the bending moment diagram:

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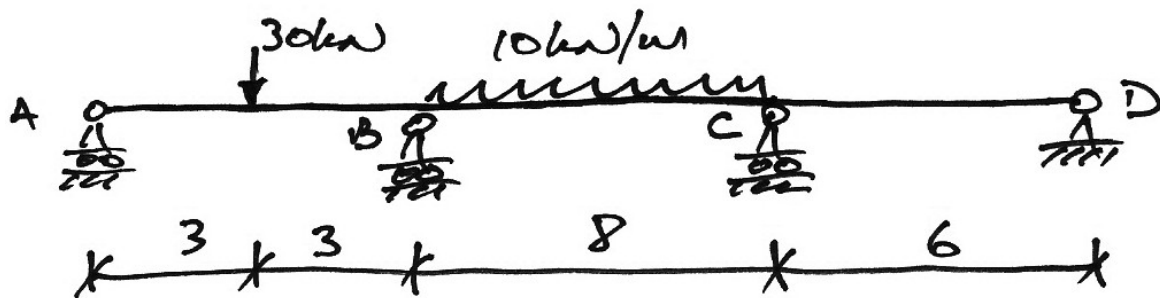
6. Virtual Work for Self-Stressed Structures

6.1 Background

Introduction

Self-stressed structures are structures that have stresses induced, not only by external loading, but also by any of the following:

- Temperature change of some of the members (e.g. solar gain);
- Lack of fit of members from fabrication:
    - Error in the length of the member;
    - Ends not square and so a rotational lack of fit;
- Incorrect support location from imperfect construction;
- Non-rigid (i.e. spring) supports due to imperfect construction.

Since any form of fabrication or construction is never perfect, it is very important for us to know the effect (in terms of bending moment, shear forces etc.) that such errors, even when small, can have on the structure.

Here we introduce these sources, and examine their effect on the virtual work equation. Note that many of these sources of error can exist concurrently. In such cases we add together the effects from each source.

Temperature Change

The source of self-stressing in this case is that the temperature change causes a member to elongate:

$$\Delta L_T = \alpha L (\Delta T)$$

where α is the coefficient of linear thermal expansion (change in length, per unit length, per degree Celsius), L is the original member length and ΔT is the temperature change.

Since temperature changes change the length of a member, the internal virtual work is affected. Assuming a truss member is being analysed, we now have changes in length due to force and temperature, so the total change in length of the member is:

$$e = \frac{PL}{EA} + \alpha L (\Delta T)$$

Hence the internal virtual work for this member is:

$$\delta W_I = e \cdot \delta P$$
$$= \left(\frac{PL}{EA} + \alpha L (\Delta T) \right) \cdot \delta P$$

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Linear Lack of Fit

For a linear lack of fit, the member needs to be artificially elongated or shortened to fit it into place, thus introducing additional stresses. This is denoted:

$$\lambda_L$$

Considering a truss member subject to external loading, the total change in length will be the deformation due to loading and the linear lack of fit:
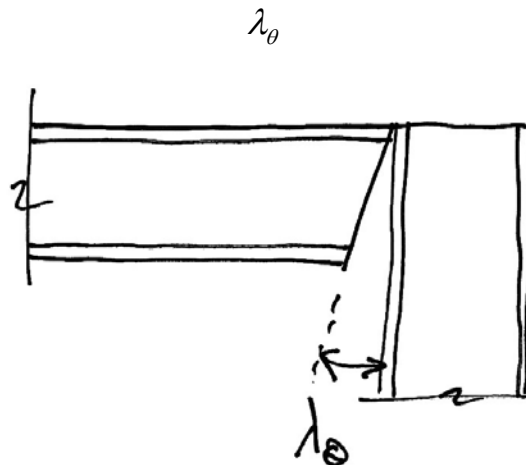
$$e = \frac{PL}{EA} + \lambda_L$$

Hence the internal virtual work for this member is:

$$\delta W_I = e \cdot \delta P$$
$$= \left(\frac{PL}{EA} + \lambda_L \right) \cdot \delta P$$

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Rotational Lack of Fit

A rotational lack of fit, which applies to frames only, occurs when the end of a member is not square. Thus the member needs to be artificially rotated to get it into place, as shown below. This is denoted as:

Considering a frame member which has a lack of fit, λ_θ and a virtual moment δM at the same point, then the internal virtual work done at this point is:

$$\delta W_I = \lambda_\theta \cdot \delta M$$

This must be added to the other forms of internal virtual work. Not also that the signs must be carefully chosen so that the virtual moment closes the gap – we will see this more clearly in an example.

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Errors in Support Location

The support can be misplaced horizontally and/or vertically. It is denoted:

$$\lambda_S$$

A misplaced support affects the external movements of a structure, and so contributes to the external virtual work. Denoting the virtual reaction at the support, in the direction of the misplacement as δR, then we have:

$$\delta W_e = \lambda_S \cdot \delta R$$

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Spring Supports

For spring supports we will know the spring constant for the support, denoted:

$$k_S$$

Since movements of a support are external, spring support movements affect the external virtual work. The real displacement Δ_S that occurs is:

$$\Delta_S = R/k_S$$

In which R is the real support reaction in the direction of the spring. Further, since R will be known in terms of the multiplier and virtual reaction, δR, we have:

$$R = R^0 + \alpha \cdot \delta R$$

Hence:

$$\Delta_S = \left(R^0 + \alpha \cdot \delta R\right)/k_S$$

And so the external work done is:

$$\delta W_e = \Delta_S \cdot \delta R$$
$$= \frac{R^0}{k_S} \cdot \delta R + \alpha \cdot \frac{(\delta R)^2}{k_S}$$

The only unknown here is α which is solved for from the virtual work equation.
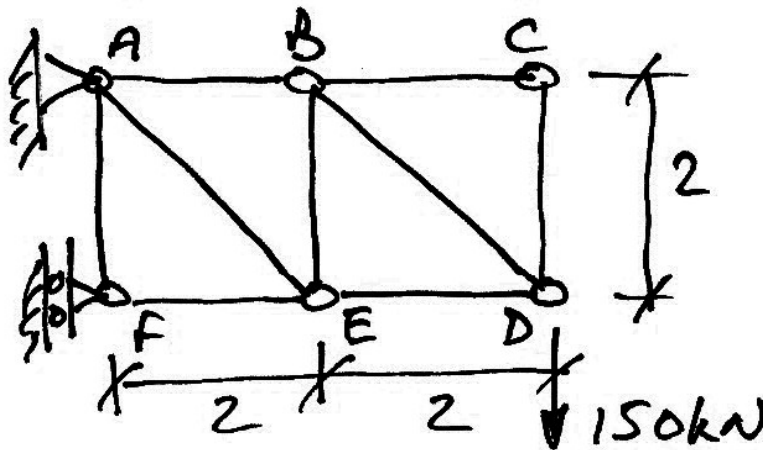
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6.2 Trusses

Example 14

Here we take the truss of Example 6 and examine the effects of:

- Member *ED* was found to be 5 mm too long upon arrival at site;
- Member *AB* is subject to a temperature increase of $+100$ ˚C.

For this truss, $E = 200$ kN/mm^2; member areas, $A = 1000$ mm^2 for all members except *AE* and *BD* where $A = 1000\sqrt{2}$ mm^2, and; coefficient of expansion is $\alpha = 10 \times 10^{-6}$ ˚C^{-1}.

The change in length due to the temperature change is:

$$\Delta L_T = \alpha L (\Delta T)$$
$$= (10 \times 10^{-6})(2000)(+100)$$
$$= 2 \text{ mm}$$

Vertical Displacement of Joint *D*

The virtual work equation is now:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$
$$y_{DV} \cdot 1 = \sum \left(\frac{P^0 L}{EA} \right)_i \cdot \delta P_i^1 + 5 \cdot \delta P_{ED}^1 + 2 \cdot \delta P_{AB}^1$$

In Example 6 we established various values in this equation:

- $\sum \left(\dfrac{P^0 L}{EA} \right)_i \cdot \delta P_i^1 = 16.5$

- $\delta P_{ED}^1 = -1$

- $\delta P_{AB}^1 = +1$

Hence we have:

$$y_{DV} \cdot 1 = 16.5 + 5(-1) + 2(+1)$$
$$y_{DV} = 13.5 \text{ mm}$$

Horizontal Displacement of Joint D:

Similarly, copying values from Example 6, we have:

$$\delta W = 0$$
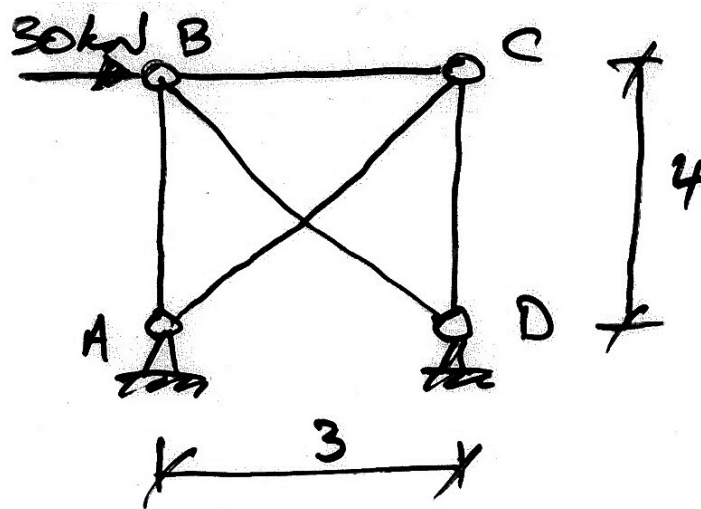$$\delta W_E = \delta W_I$$
$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$
$$y_{DH} \cdot 1 = \sum \left(\frac{P^0 L}{EA} \right)_i \cdot \delta P_i^2 + 5 \cdot \delta P_{ED}^2 + 2 \cdot \delta P_{AB}^2$$
$$= -4.5 + 5(+1) + 2(0)$$
$$y_{DH} = +0.5 \text{ mm to the right}$$

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Example 15

Here we use the truss of Example 11 and examine, separately, the effects of:

- Member *AC* was found to be 3.6 mm too long upon arrival on site;
- Member *BC* is subject to a temperature reduction of -50 ˚C;
- Support *D* is surveyed and found to sit 5 mm too far to the right.

Take *EA* to be 10×10^4 kN for all members and the coefficient of expansion to be $\alpha = 10 \times 10^{-6}$ ˚C^{-1}.

Error in Length:

To find the new multiplier, we include this effect in the virtual work equation:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$
$$0 = \sum \left(\frac{PL}{EA} \right)_i \cdot \delta P_i^1 + \lambda_L \cdot \delta P_{AC}^1$$

But $P = P^0 + \alpha \cdot \delta P^1$, hence:

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$$0 = \sum \left(\frac{\left(P^0 + \alpha \cdot \delta P^1\right) L}{EA} \right)_i \cdot \delta P_i^1 + \lambda_L \cdot \delta P_{AC}^1$$

$$= \sum \left(\frac{P^0 L}{EA} \right)_i \cdot \delta P_i^1 + \alpha \cdot \sum \left(\frac{\delta P^1 L}{EA} \right)_i \cdot \delta P_i^1 + \lambda_L \cdot \delta P_{AC}^1$$

$$0 = \sum \frac{P^0 \cdot \delta P_i^1 \cdot L_i}{EA_i} + \alpha \cdot \sum \frac{\left(\delta P_i^1\right)^2 L_i}{EA_i} + \lambda_L \cdot \delta P_{AC}^1$$

As can be seen, the $\lambda_L \cdot \delta P_{AC}^1$ term is simply added to the usual VW equation. From before, we have the various values of the summations, and so have:

$$0 = \frac{-37.8 \times 10^4}{EA} + \alpha \cdot \frac{1.62 \times 10^4}{EA} + (3.6)(+1)$$

Hence:

$$\alpha = \frac{37.8 \times 10^4 - 3.6 EA}{1.62 \times 10^4} = +1.11$$

Thus member *AC* is 1.1 kN in tension. Note the change: without the error in fit it was 23.3 kN in tension and so the error in length has reduced the tension by 22.2 kN.

Temperature Change:

The same derivation from the VW equation gives us:

$$0 = \sum \frac{P^0 \cdot \delta P_i^1 \cdot L_i}{EA_i} + \alpha \cdot \sum \frac{\left(\delta P_i^1\right)^2 L_i}{EA_i} + \Delta L_T \cdot \delta P_{BC}^1$$

The change in length due to the temperature change is:

$$\Delta L_T = \alpha L (\Delta T)$$
$$= (10 \times 10^{-6})(3000)(-50)$$
$$= -1.5 \text{ mm}$$

Noting that the virtual force in member BC is $\delta P_{BC}^1 = 0.6$, we have:

$$0 = \frac{-37.8 \times 10^4}{EA} + \alpha \cdot \frac{1.62 \times 10^4}{EA} + (-1.5)(0.6)$$

Giving:

$$\alpha = \frac{37.8 \times 10^4 + 0.9EA}{1.62 \times 10^4} = +28.9$$

Thus member AC is 28.9 kN in tension, an increase of 5.4 kN.

Error in Support Location:

Modifying the VW equation gives:

$$\lambda_S \cdot \delta H_D = \sum \frac{P^0 \cdot \delta P_i^1 \cdot L_i}{EA_i} + \alpha \cdot \sum \frac{\left(\delta P_i^1\right)^2 L_i}{EA_i}$$

The value of the virtual horizontal reaction is found from the virtual force system to be 0.6 kN to the right. Hence:

$$+(5)(0.6) = \frac{-37.8 \times 10^4}{EA} + \alpha \cdot \frac{1.62 \times 10^4}{EA}$$

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Note the sign on the support displacement: since the real movement is along the same direction as the virtual force, it does positive virtual work. Solving:

$$\alpha = \frac{-3EA + 37.8 \times 10^4}{1.62 \times 10^4} = +4.8$$

Thus member AC is 4.8 kN in tension; a reduction of 18.5 kN.

All Effects Together:

In this case, the virtual work equation is:

$$\lambda_S \cdot \delta H_D = \sum \frac{P^0 \cdot \delta P_i^1 \cdot L_i}{EA_i} + \alpha \cdot \sum \frac{\left(\delta P_i^1\right)^2 L_i}{EA_i} + \lambda_L \cdot \delta P_{AC}^1 + \Delta L_T \cdot \delta P_{BC}^1$$

Substituting the various values in gives:

$$+(0.5)(0.6) = \frac{-37.8 \times 10^4}{EA} + \alpha \cdot \frac{1.62 \times 10^4}{EA} + (3.6)(+1) + (-1.5)(0.6)$$

And solving:

$$\alpha = \frac{(-3.6 + 0.9 - 3)EA + 37.8 \times 10^4}{1.62 \times 10^4} = -11.9$$

And so member AC is 11.9 kN in compression. Thus should be the same as the original force of 23.3 kN plus the changes induced by the errors:

$$23.3 - 22.2 + 5.4 - 18.5 = -12 \text{ kN}$$

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6.3   Beams

Example 16

Consider the following 2-span beam with central spring support. Determine an expression for the central support reaction in terms of its spring stiffness.
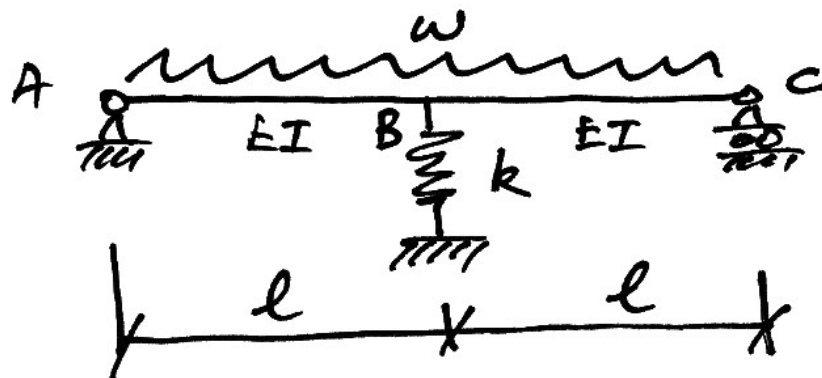
To do this, we will consider the central support as the redundant:
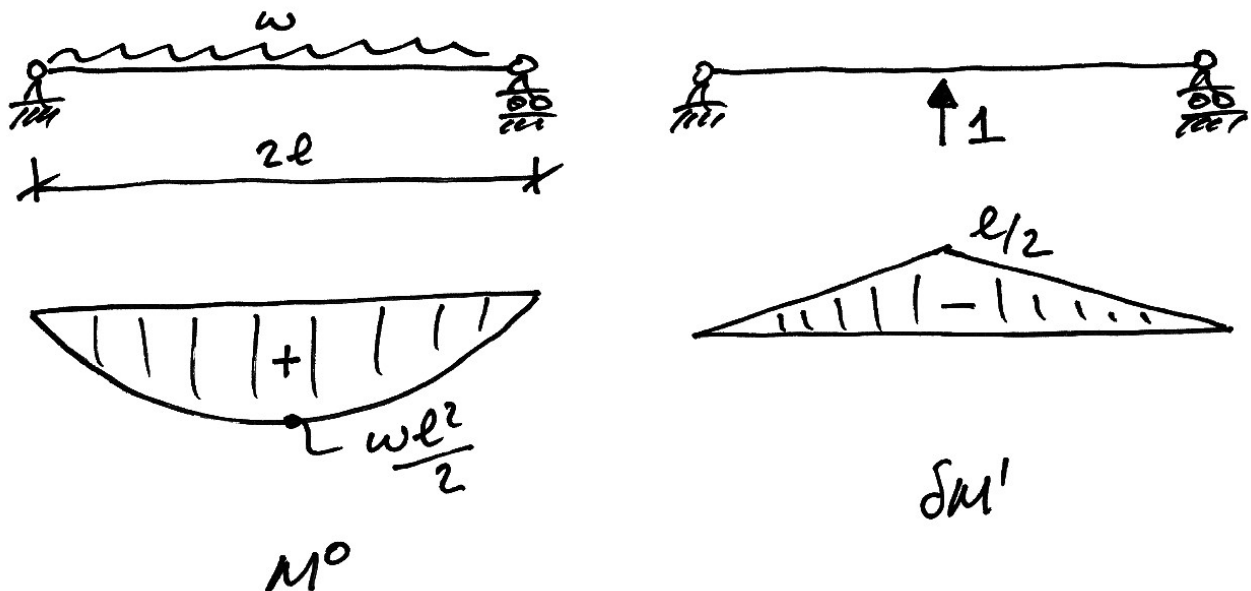
The virtual work equation, accounting for the spring, is:

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$$1 \cdot \Delta_S = \sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx + \alpha \cdot \sum \int_0^L \frac{\left(\delta M_i^1\right)^2}{EI_i} dx$$

The deflection at the central support will be:

$$\Delta_S = \frac{V_B}{k} = \frac{\alpha}{k}$$

Since the reaction at B is $1 \cdot \alpha$. Noting that the deflection of the spring will be opposite to the unit load, and using the volume integrals, we have:

$$-\frac{\alpha}{k} = \frac{2}{EI}\left[\frac{5}{12}\left(-\frac{l}{2}\right)\left(\frac{wl^2}{2}\right)(l)\right] + \alpha \cdot \frac{2}{EI}\left[\frac{1}{3}\left(-\frac{l}{2}\right)\left(-\frac{l}{2}\right)(l)\right]$$

$$-\frac{\alpha}{k} = \frac{-5wl^4}{24EI} + \alpha \cdot \frac{l^3}{6EI}$$

$$\alpha\left(-\frac{EI}{k} - \frac{l^3}{6}\right) = \frac{-5wl^4}{24}$$

And finally:

$$\alpha = \frac{5wl^4}{\dfrac{24EI}{k} + 4l^3}$$

$$= \frac{5wl}{24\dfrac{EI}{kl^3} + 4}$$

It is important we understand the implications of this result:

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- For no support present, $k = 0$ and so $24EI/k \to \infty$ meaning $\alpha \to 0$ and there is no support reaction (as we might expect);

- For $k = \infty$ we have the perfectly rigid (usual) roller support and so $24EI/k \to 0$ giving us $\alpha \to \dfrac{5}{4}wl$ - a result we established previously;

- For intermediate relative stiffnesses, the value of the reaction is between these extremes. Therefore, we as the designer have the ability to choose the reaction most favourable to us.

A plot of the reaction and relative stiffnesses is:

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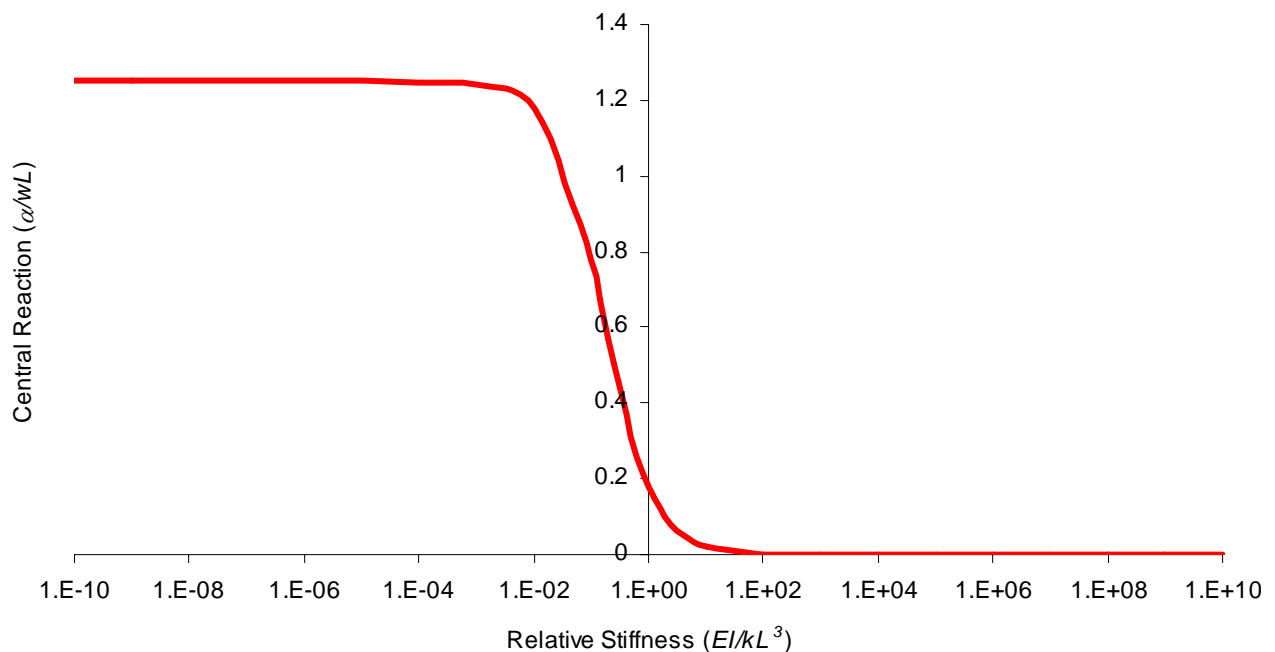
6.4   Frames

Example 17

The following frame, in addition to its loading, is subject to:

- Support A is located 10 mm too far to the left;

- End C of member BC is 1.2×10^{-3} rads out of square, as shown;

- Member CD is 12 mm too short.

Determine the bending moment diagram. Take $EI = 36 \times 10^3$ kNm^2 for each member.
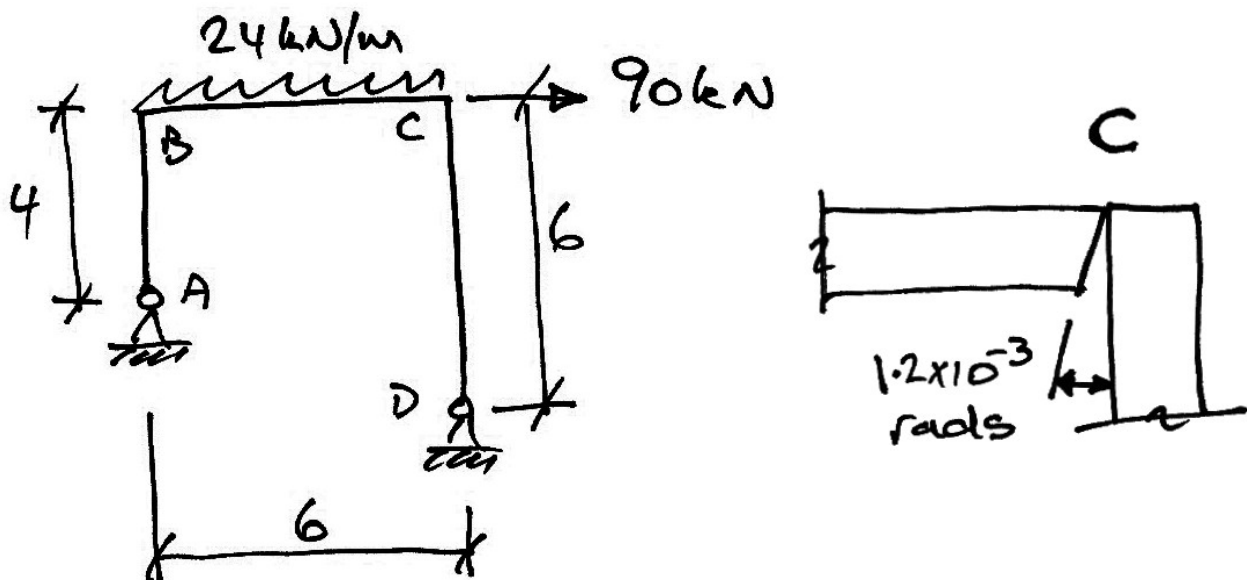
We will choose the horizontal reaction at A as the redundant. Since we are dealing with a linear lack of fit in member CD, we need to allow for the virtual work done by the axial forces in this member and so we solve for the axial force diagrams also.

For the primary structure:

$$\sum F_x = 0: \quad H_D^0 - 90 = 0 \qquad \therefore H_D^0 = 90 \text{ kN}$$

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$\sum M$ about $A = 0$:

$$24 \cdot \frac{6^2}{2} + 90 \cdot 4 + 90 \cdot 2 - V_D^0 \cdot 6 = 0 \qquad \therefore V_D^0 = 162 \text{ kN}$$

$$\sum F_y = 0: \quad -V_A^0 - 24 \cdot 6 + 162 = 0 \qquad \therefore V_A^0 = 18 \text{ kN}$$

Also, $M_C = 90 \cdot 6 = 540$ kNm giving:
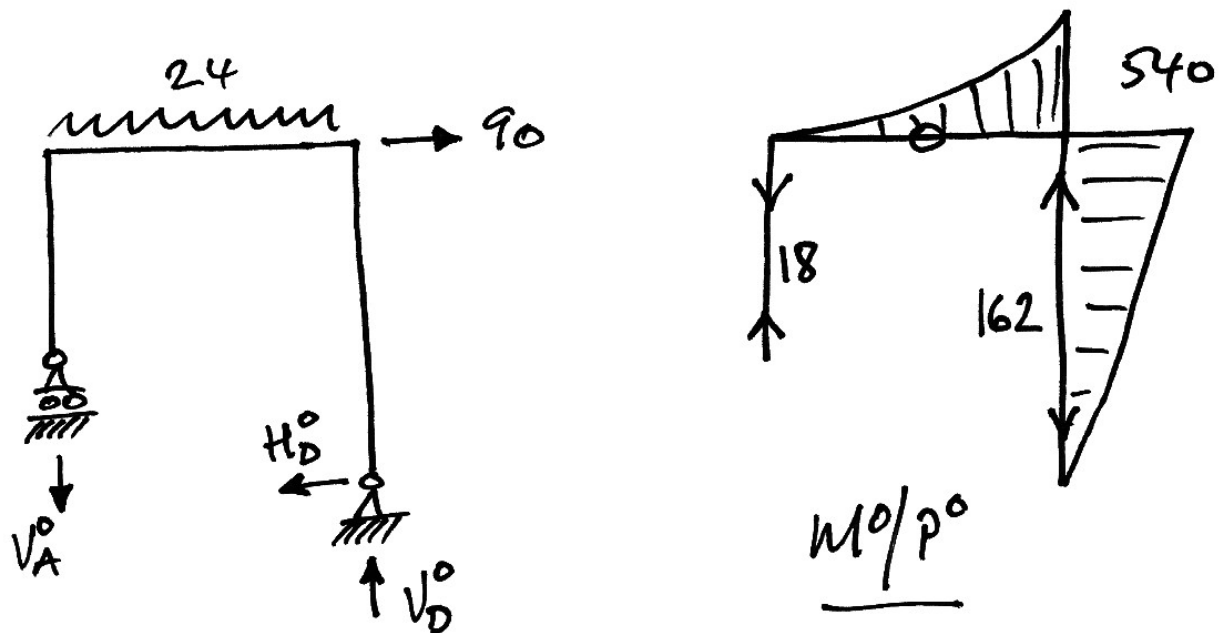
For the redundant structure:

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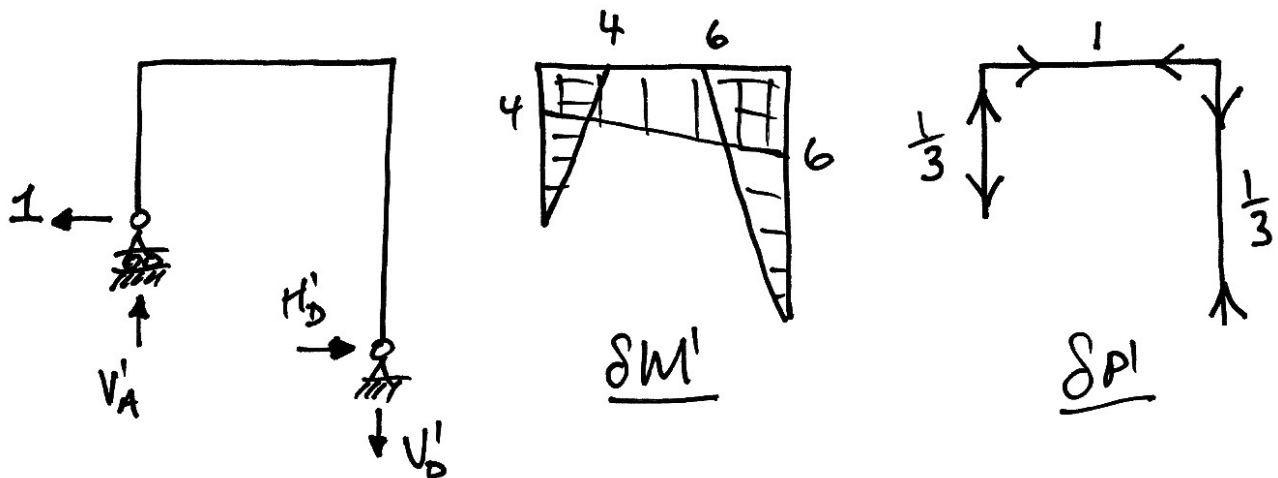
$$\sum M \text{ about } A = 0: \ -1 \cdot 2 + V_D^1 \cdot 6 = 0 \qquad \therefore V_D^1 = \frac{1}{3} \text{ kN} \downarrow$$

$$\sum F_y = 0: \quad -V_A^1 + \frac{1}{3} = 0 \ \ \therefore V_A^1 = \frac{1}{3} \text{ kN} \uparrow$$

The virtual work equation, accounting for the relevant effects is:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\lambda_S \cdot H_A^1 = \sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx + \alpha \cdot \sum \int_0^L \frac{\left(\delta M_i^1\right)^2}{EI_i} dx + \lambda_\theta \cdot \delta M_C^1 + \lambda_L \cdot \delta P_{CD}^1$$

We take each term in turn:

(a) $\lambda_S \cdot H_A^1$: The applied unit load is in the same direction as the error in the support location, and keeping all units in metres:

$$\lambda_S \cdot H_A^1 = 10 \times 10^{-3} \cdot +1 = 10 \times 10^{-3} \ \left(\text{kN} \cdot \text{m}\right)$$

(b) $\sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx$: Using the volume integrals:

$$\sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx = \frac{1}{EI} \left\{ \left[\frac{1}{12}(4 + 3 \cdot 6)(-540)6 \right]_{BC} + \left[\frac{1}{3}(6)(-540)6 \right]_{CD} \right\}$$

$$= -\frac{12420}{EI}$$

(c) $\sum \int_0^L \frac{\left(\delta M_i^1\right)^2}{EI_i} dx$: Again using the integrals table:

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$$\sum_0^L \frac{\left(\delta M_i^1\right)^2}{EI_i} dx = \frac{1}{EI}\left[\frac{1}{3}(4)(4)4\right]_{AB} + \frac{1}{EI}\left[\frac{1}{6}\left(4(2\cdot 4 + 6) + 6(4 + 2\cdot 6)\right)6\right]_{BC}$$

$$+ \frac{1}{EI}\left[\frac{1}{3}(6)(6)6\right]_{CD}$$

$$\sum_0^L \frac{\left(\delta M_i^1\right)^2}{EI_i} dx = \frac{245.33}{EI}$$

(d) $\lambda_\theta \cdot \delta M_C^1$: The virtual bending moment at C is 6 kNm with tension on the inside of the frame. This is in the opposite direction to that needed to close the lack of fit and so the sign of this term is negative, as shown:
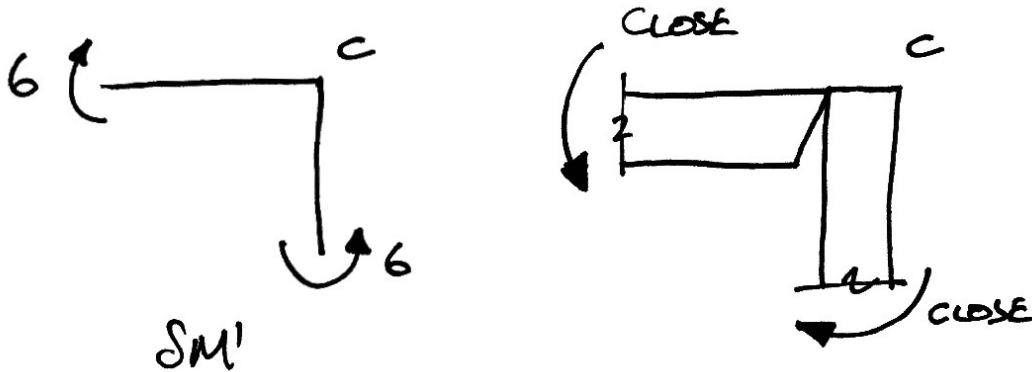
$$\lambda_\theta \cdot \delta M_C^1 = -\left(1.2\times10^{-3}\cdot 6\right) = -7.2\times10^{-3}\ (kN\cdot m)$$

(e) $\lambda_L \cdot \delta P_{CD}^1$: Since the member is too short, the lack of fit is negative, whilst the virtual force is in tension and so positive:

$$\lambda_L \cdot \delta P_{CD}^1 = \left(-12\times10^{-3}\right)\cdot\left(+\frac{1}{3}\right) = -4\times10^{-3}\ (kN\cdot m)$$

Substituting these values into the equation, along with $EI = 36\times10^3$ kNm^2, gives:

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$$10 \times 10^{-3} = -\frac{12420}{36 \times 10^3} + \alpha \cdot \frac{245.33}{36 \times 10^3} - 7.2 \times 10^{-3} - 4 \times 10^{-3}$$

And so we can solve for α:

$$10 = -345 + \alpha \cdot 6.815 - 7.2 - 4$$
$$\alpha = \frac{10 + 345 + 7.2 + 4}{6.815}$$
$$= 53.73$$

And so the horizontal reaction at A is:

$$H_A = H_A^0 + \alpha \cdot H_A^1 = 0 + 53.73 \cdot 1 = 53.73 \text{ kN}$$

and similarly for the other reactions. Also, using the superposition equation for moments, $M = M^0 + \alpha \cdot M^1$, we have:

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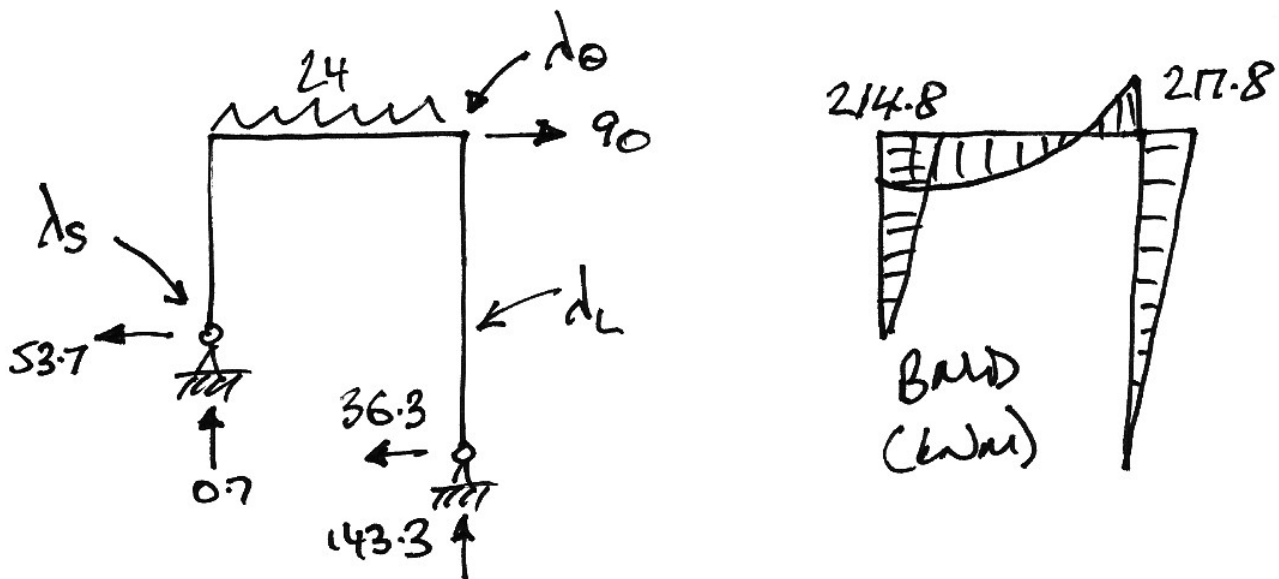
6.5 Problems

1. For the truss of Example 15, show that the force in member AC is 22.2 kN in compression when there is no external loads present, and only the lack of fit of 3.6 mm of member AC.

2. For the truss of Example 15, determine the horizontal deflection of joint C due to each of the errors separately, and then combined.

3. For the beam of Example 16, show that the stiffness of the spring support that optimizes the bending moments is $89.5\,EI/l^3$, i.e. makes the sagging and hogging moments equal in magnitude.

4. For the following beam, show that the vertical deflection at C can be given by:

$$\delta_{Cy} = P\left(\frac{2L^3}{3EI} + \frac{1}{k}\right)$$

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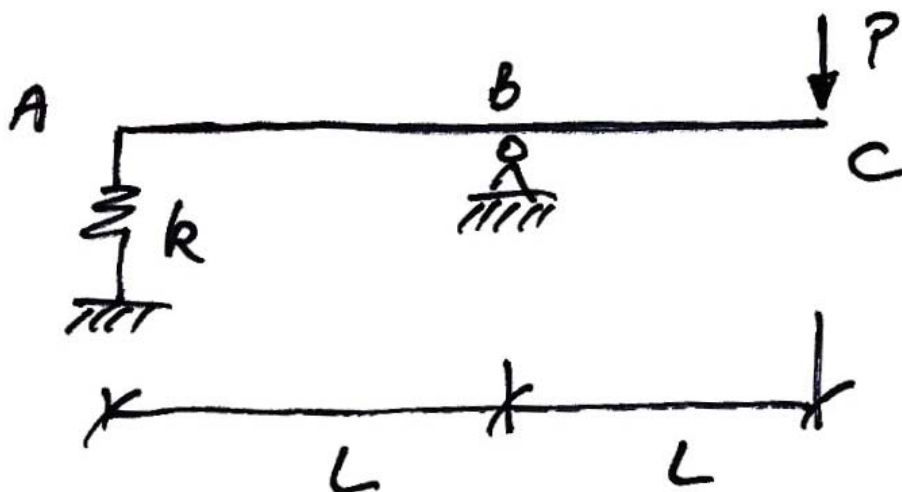
5. For the frame of Example 17, show that the horizontal deflection of joint *C* due to the applied loads only is 77.5 mm. Find the deflection of joint *C* due to both the loads and errors given.

6. For the following frame, the support at *D* was found to yield horizontally by 0.08 mm/kN. Also, the end of member *BC* is not square, as shown. Draw the bending moment diagram and determine the horizontal deflection at *D*. Take $EI = 100 \times 10^3$ kNm^2. (*Ans.* 5.43 mm to the right)

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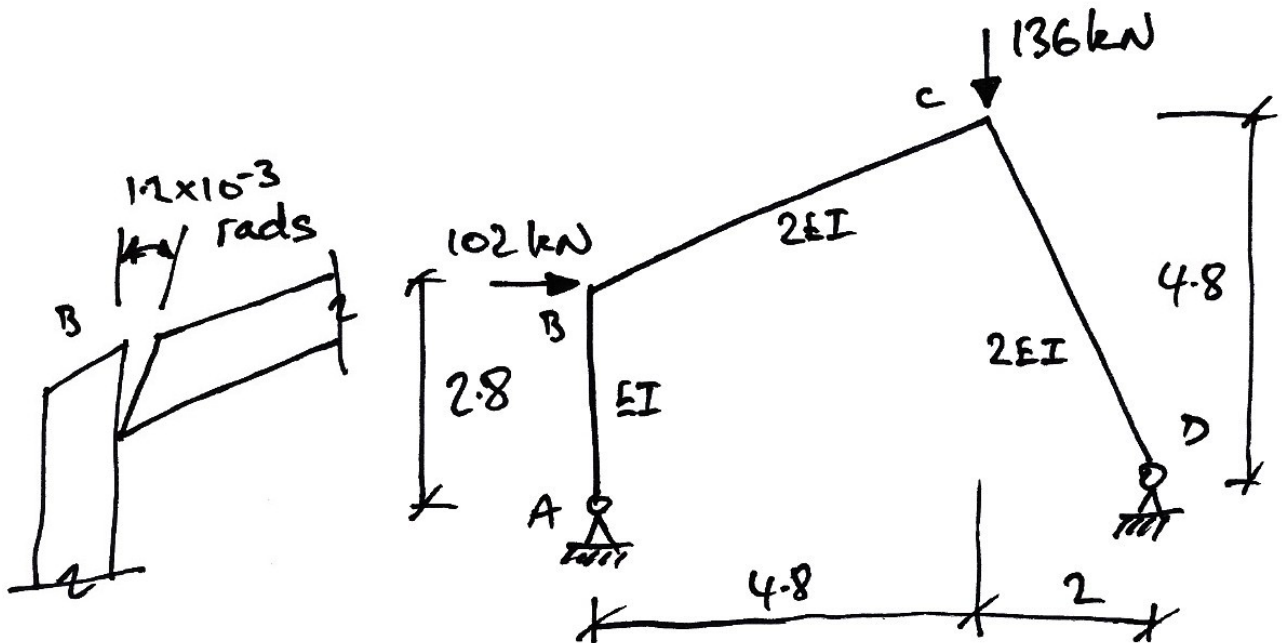
7. Past Exam Questions

7.1 Summer 1997

3. (a) Calculate the *horizontal* deflection of joint C in the pin-jointed truss shown in Fig. Q3(a) when a load of 100 kN is applied as shown at B.

   (b) If the truss were constructed with an additional member BD as shown in Fig. Q3(b), calculate the *horizontal* deflection of joint C when the load of 100 kn is applied, as before, at B.

   (c) What "initial lack of fit" in member BD will give *zero* horizontal deflection at C when the load of 100 kN is applied, as before, at B in the truss in Fig. Q2(b).

   Assume $E = 200$ kN/mm^2.

   Assume the cross-sectional of AB, BC and CD is 100 mm^2, and AC and BD is 141 mm^2.   (25 marks)
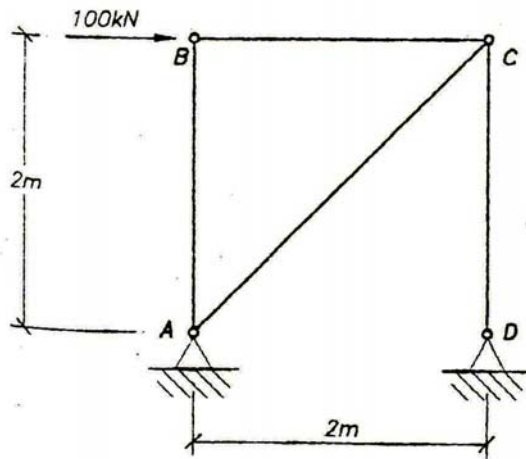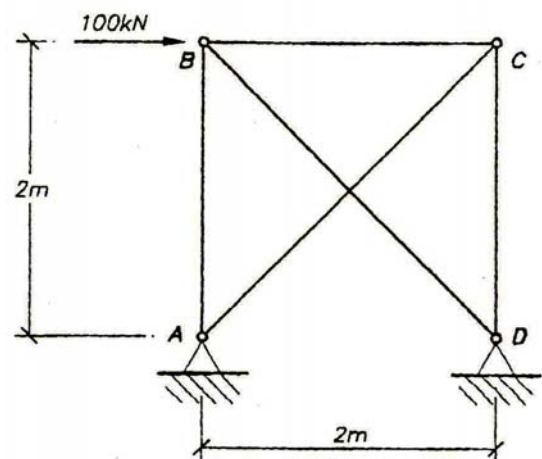
FIG Q3(a)   FIG Q3(b)

Answers:

(a) $\delta_{Cx} = 30.0$ mm $\rightarrow$;

(b) $\delta_{Cx} = 12.9$ mm $\rightarrow$;

(c) $\lambda_{BD} = 21.1$ mm too long.

7.2 Summer 1998

3. (a) Calculate the *horizontal* deflection of joint C in the pin-jointed truss loaded as shown in Fig. Q3 (a).
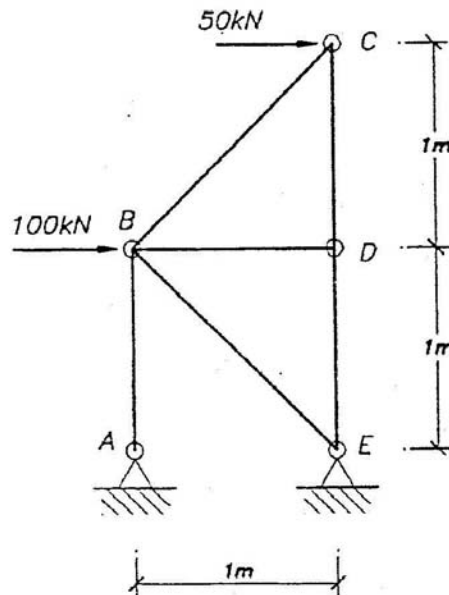
(b) If the truss were propped with an additional member DF as shown in Fig. Q3 (b), calculate the new *horizontal* deflection of joint C when the truss is loaded, as before.

Assume E = 200 kN/mm^2.
Assume the cross-sectional area of AB, BD, CD and DE is 100mm^2, and BC, BE and DF is 141mm^2.

(25 marks)
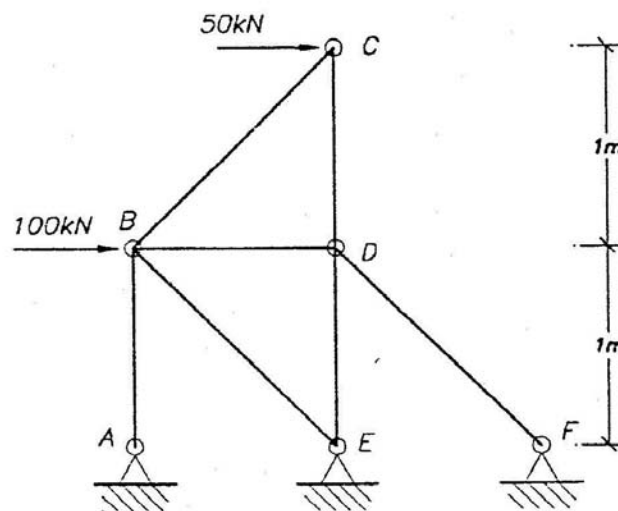
Answers:

(a) $\delta_{Cx} = 45.0$ mm $\rightarrow$;

(b) $\delta_{Cx} = 26.2$ mm $\rightarrow$.

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7.3 Summer 1999

3. Use the *Flexibility method* to determine, for the frame shown in Fig. Q3,

    (a)   the bending moment in the frame and

    (b)   the vertical deflection at C.

Sketch the bending moment diagram giving the value of the bending moment at all salient points.

Sketch the deflected shape of the frame.

Take constant EI = $12 \times 10^4 \text{kNm}^2$.   (25 marks)
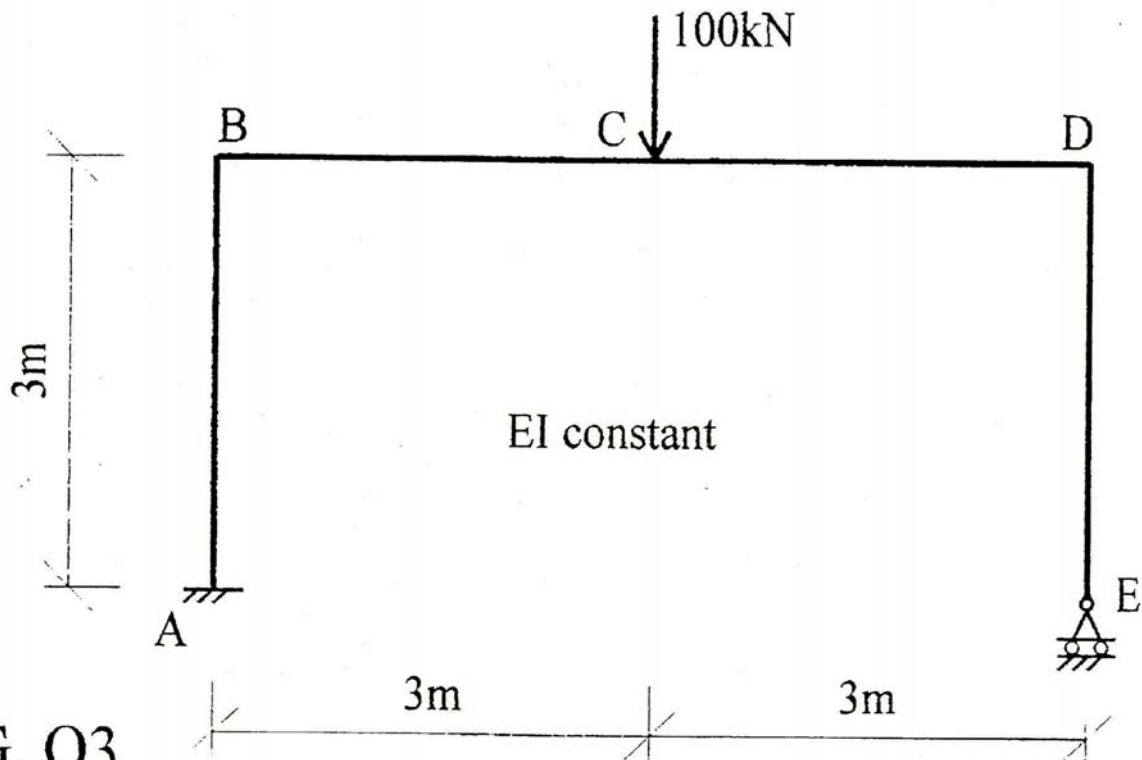
FIG. Q3

Answers:

(a) $V_E = 42.5 \text{ kN} \uparrow$;

(b) $\delta_{Cy} = 28.6 \text{ mm} \downarrow$.

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7.4 Summer 2000

3. Use the *Flexibility method* to determine, for the frame shown in Fig. Q3,

   (a) the bending moment in the frame and

   (b) the vertical deflection at C.

   Sketch the bending moment diagram giving the value of the bending moment at all salient points.

   Sketch the deflected shape of the frame.

   Take constant $EI = 12 \times 10^4 kNm^2$.

   (25 marks)
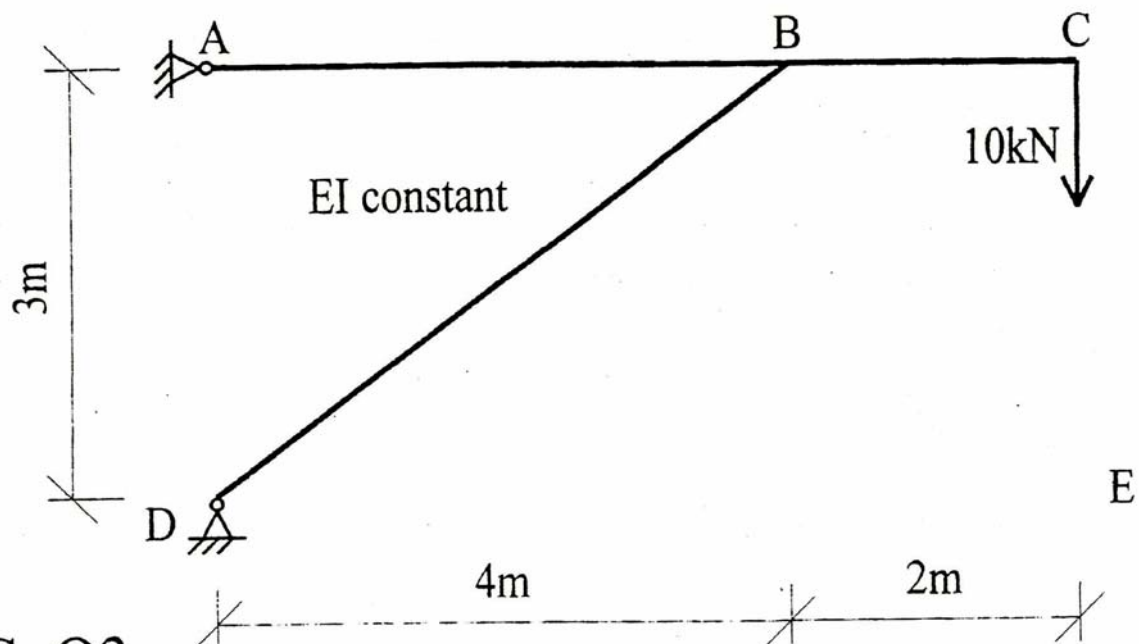
FIG. Q3

Answers:

(a) $V_A = 2.78$ kN $\downarrow$;

(b) $\delta_C = 0.47$ mm $\downarrow$.

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7.5   Summer 2001

3. (a)   On assembly of the *pin-jointed truss* shown in Fig. Q3 it was found that member AB was 4 mm
too long and support E was 10 mm too high.
The truss was then loaded as shown at B. Find the forces in the members.   (17 marks)

(b)   If *in addition* to all the conditions in (a) above, (i.e. AB 4 mm too long, support E 100 mm too
high, and the loading as shown), it is found that support A yields to the right for each 1 kN of
reaction at support A, determine the forces in the members.   (8 marks)

Assume EA = 120×10^3 kN for all members.
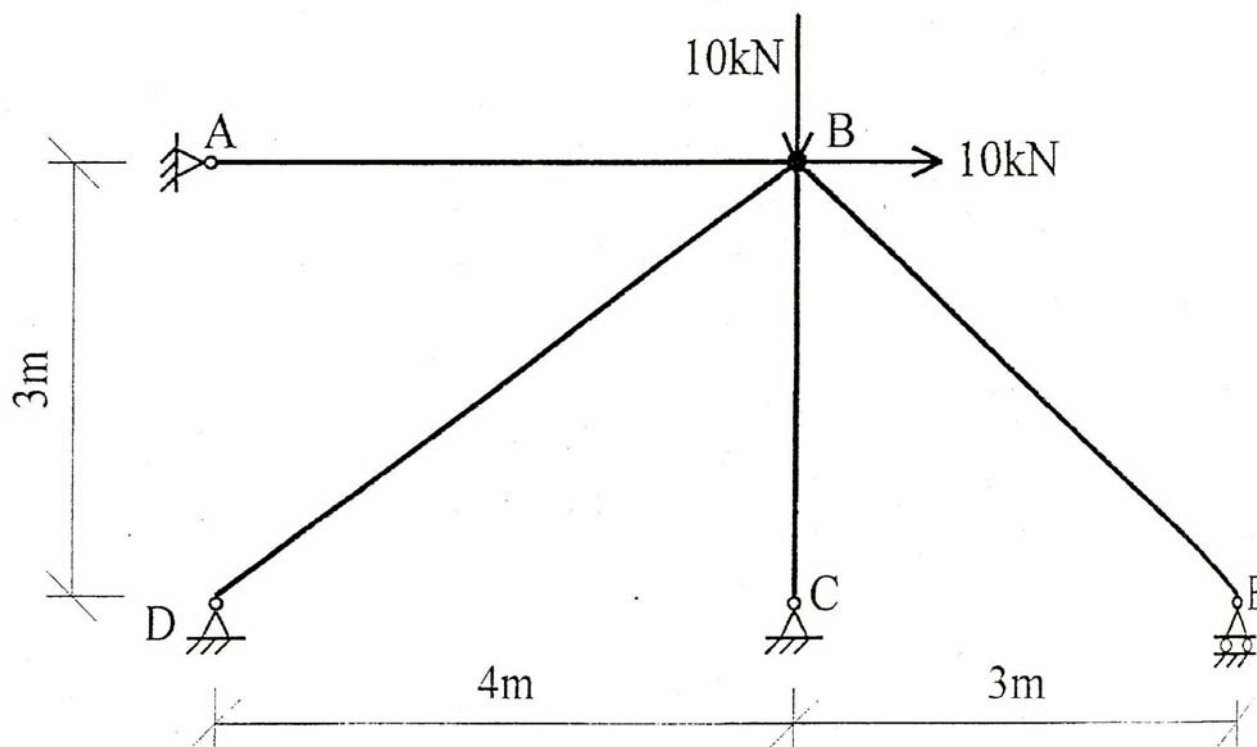
FIG. Q3

Answers:

(a) $P_{BD} = +44.4$ kN ;

(b) $P_{BD} = +300$ kN .

7.6 Summer 2002

3. (a) Using *Virtual Work* determine by what length member AC, in the truss loaded as shown in Fig. Q3 (a), must be adjusted to ensure that the vertical deflection at node C is zero. Assume $EA = 200 \times 10^3$ kN for all members.   (6 marks)
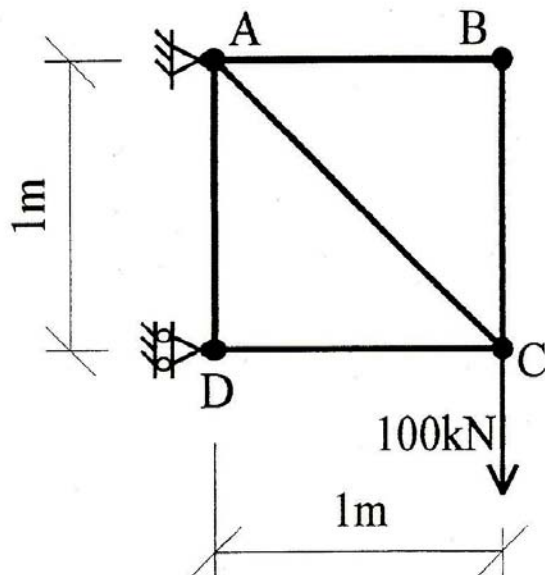
(b) (i) Using Virtual Work, and neglecting axial deformation, determine the bending moments in the frame loaded as shown in Fig. Q3(b).   (15 marks)

   (ii) Sketch the bending moment diagram for the frame.   (2 marks)

   (iii) Sketch the deflected shape of the frame.   (2 marks)
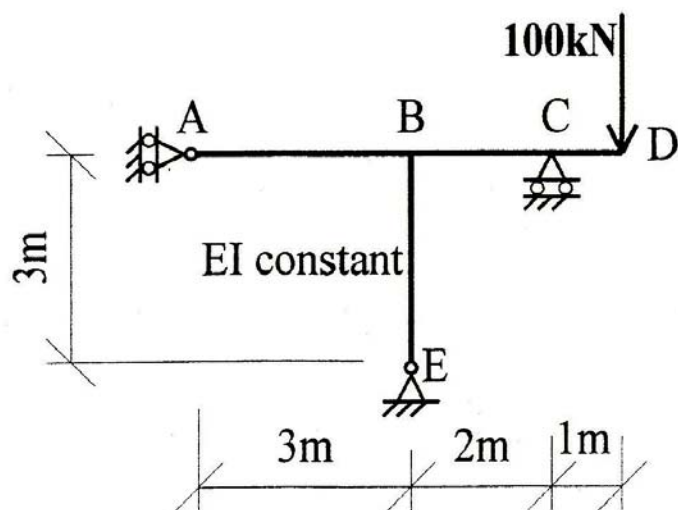
Answers:

(a) $\lambda_L = 1.35$ mm shorter;

(b) $V_C = 160$ kN $\uparrow$.

7.7   Summer 2004

3. (a)   Calculate the vertical deflection of joint A in the pin-jointed truss shown in Fig. Q3 when a load of 100 kN is applied at A as shown.
(20 marks)

(b)   What initial lack of fit in member *AC* will give zero vertical deflection at *A* when the load of 100 kN is applied, as before, at *A* in the truss in Fig. Q3.
(5 marks)

Assume $E = 200$ kN/mm^2.
Assume the cross-sectional area of *AB* and *AD* is 500 mm^2 each, and of *AC* is 400 mm^2.
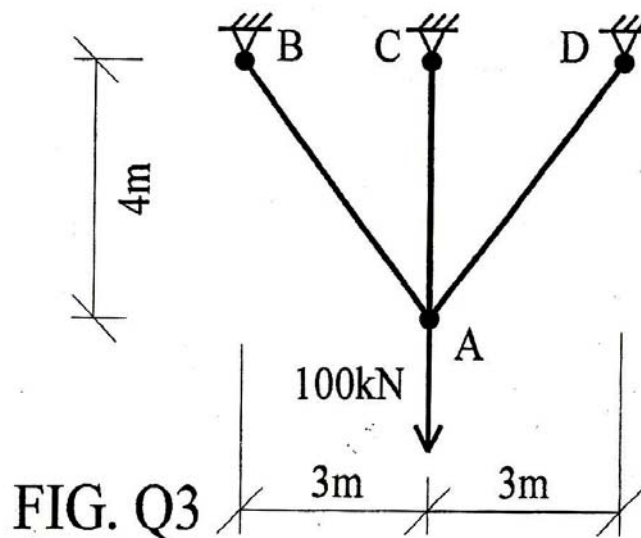
FIG. Q3

Answers:

(a) $\delta_{Ay} = 2.19$ mm $\downarrow$;

(b) $\lambda_L = 5$ mm shorter.

7.8 Summer 2007

Using *Virtual Work*:

(i) For the frame in Fig. Q3(a), determine the horizontal deflection of joint C;

(ii) Draw the bending moment diagram for the frame in Fig. Q3(b);

(iii) Determine the horizontal deflection of joint C for the frame in Fig. Q3(b).

Note:

You may neglect axial effects in the members.

Take $EI = 36 \times 10^3$ kNm^2 for all members.
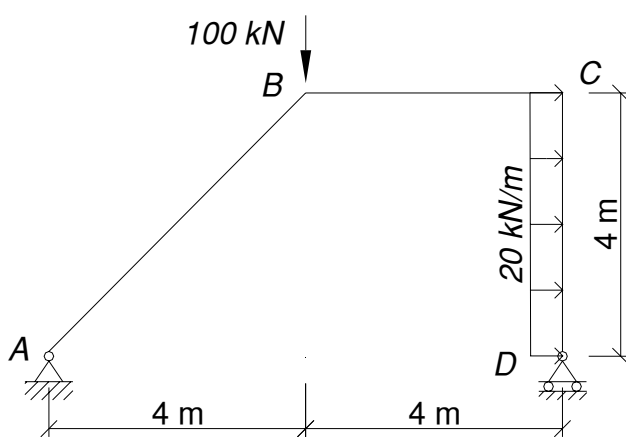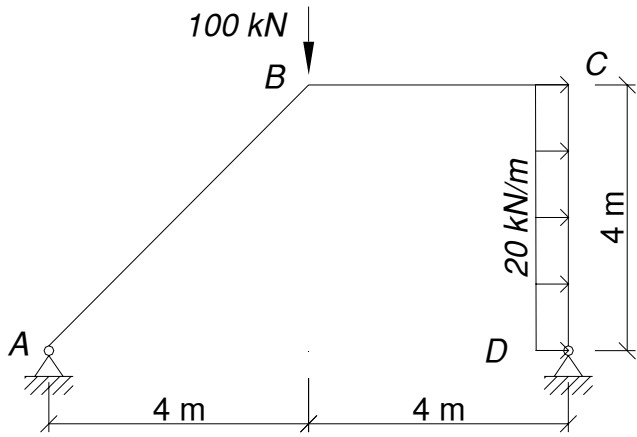
FIG. Q3(a)   FIG. Q3(b)

Answers:

(a) $\delta_{Cx} = 84.6$ mm $\rightarrow$;

(b) $H_D = 75.8$ kN $\leftarrow$;

(c) $\delta_{Cx} = 19.02$ mm $\rightarrow$.

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7.9 Semester 2, 2008

For the frame shown in Fig. Q2, using *Virtual Work*:

(a) Determine the following:
   (i)   reactions;
   (ii)   bending moment diagram.

(b) After construction, it is found that the support at *C* is not rigid but a spring support of stiffness $k = 2000/3$ kN/m.
   Taking this into account, determine the following:
   (i)   vertical displacement of joint *C*;
   (ii)   bending moment diagram.

(40 marks)

Note:

You may neglect axial effects in the members.

Take $EI = 36 \times 10^3$ kNm^2 for all members.
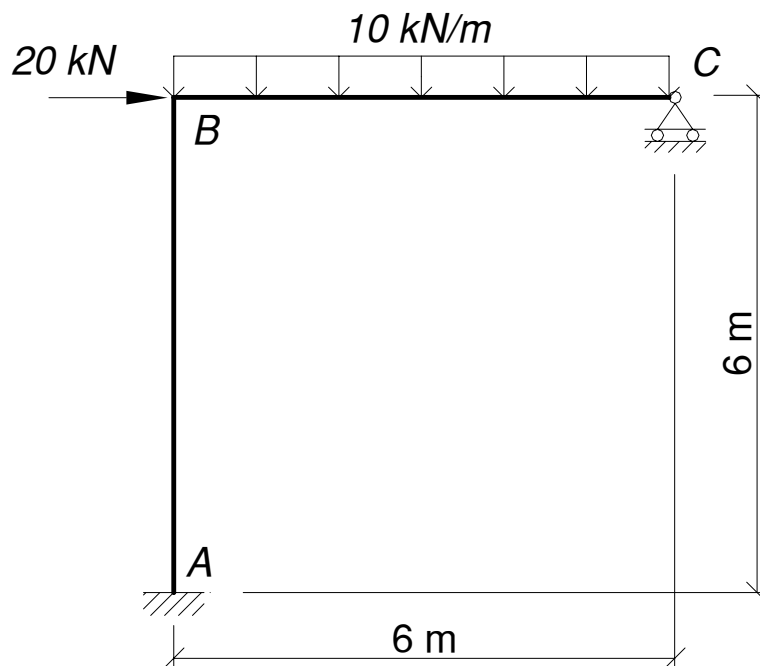
FIG. Q2

Answers:

(a) $V_C = 35.63$ kN $\uparrow$;

(b) $\delta_{Cy} = 45$ mm $\downarrow$.

7.10 Semester 2, 2009

QUESTION 2

Using *Virtual Work*, for the frame shown in Fig. Q2, do the following:

(i)   Determine the reactions;

(ii)   Draw the bending moment diagram;

(iii)   Draw the deflected shape of the structure;

(iv)   Determine the horizontal deflection of joint *D*.

(25 marks)

Note:

• Take $E = 200$ kN/mm^2 and $I = 45 \times 10^7$ mm^4 for all members.
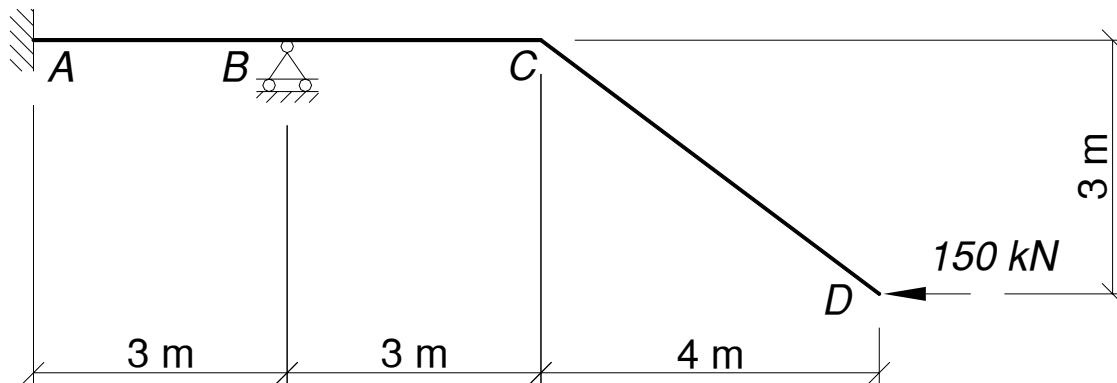
FIG. Q2

Answers:

(a) $V_B = 225$ kN $\uparrow$;

(b) $\delta_{Dx} = 81.25$ mm $\leftarrow$.

7.11 Semester 2, 2009

QUESTION 3

For the structure shown in Fig. Q2, due to an error on site, it is required to make the moment at support *A* zero. This is to be done by the introduction of a spring support at *B*, in place of the roller support, as shown in Fig. Q3. Using *Virtual Work*:

(i)   Determine the spring stiffness required so that the moment at A is zero.

(ii)   Drawn the revised bending moment diagram;

(iii)   Draw the revised deflected shape of the structure;

(iv)   Determine the vertical deflection of the structure at *B*.

(25 marks)

Note:

- Take $E = 200$ kN/mm^2 and $I = 45 \times 10^7$ mm^4 for all members.
- You may use any relevant results from your workings for Q2, but in doing so acknowledge their source.
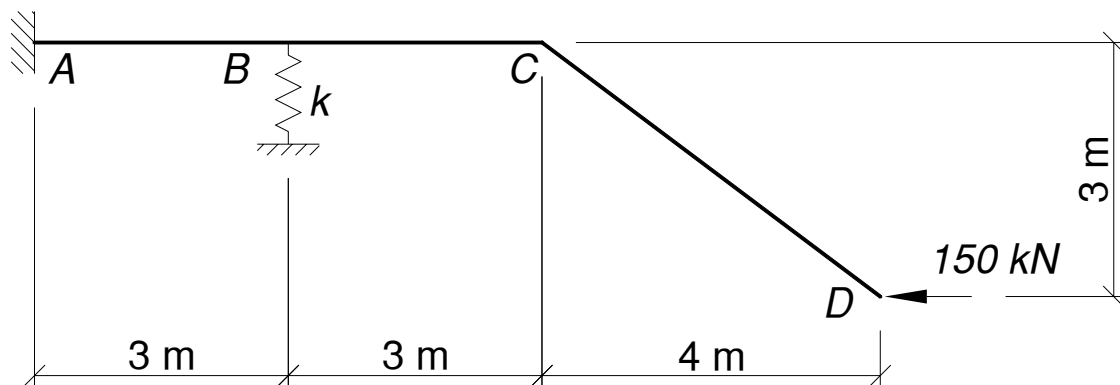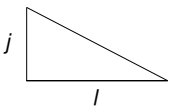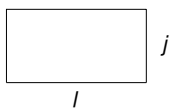

FIG. Q3
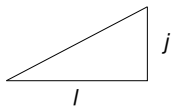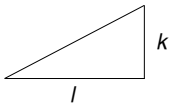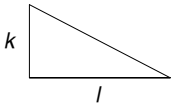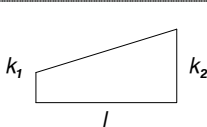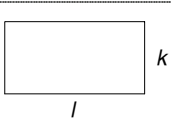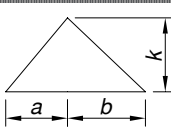
Answers:

(a) $k = 20 \times 10^3$ kN/m;

(b) $\delta_{By} = 7.5$ mm $\downarrow$.

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8. References

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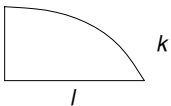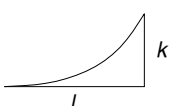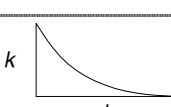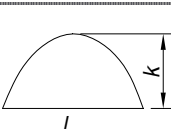- Coates, R.C., Coutie, M.G. and Kong, F.K., *Structural Analysis*, Chapman and Hall, 1987.
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9. Appendix – Volume Integrals

	j ◥ l	j ◣ l	j_1 ▱ j_2 l	j ▭ l
◢ k, l	$\dfrac{1}{3}jkl$	$\dfrac{1}{6}jkl$	$\dfrac{1}{6}\left(j_1+2j_2\right)kl$	$\dfrac{1}{2}jkl$
k ◥ l	$\dfrac{1}{6}jkl$	$\dfrac{1}{3}jkl$	$\dfrac{1}{6}\left(2j_1+j_2\right)kl$	$\dfrac{1}{2}jkl$
k_1 ▱ k_2, l	$\dfrac{1}{6}j\left(k_1+2k_2\right)l$	$\dfrac{1}{6}j\left(2k_1+k_2\right)l$	$\dfrac{1}{6}\Big[j_1\left(2k_1+k_2\right)+ j_2\left(k_1+2k_2\right)\Big]l$	$\dfrac{1}{2}j\left(k_1+k_2\right)l$
▭ k, l	$\dfrac{1}{2}jkl$	$\dfrac{1}{2}jkl$	$\dfrac{1}{2}\left(j_1+j_2\right)kl$	jkl
△ k, a b	$\dfrac{1}{6}jk\left(l+a\right)$	$\dfrac{1}{6}jk\left(l+b\right)$	$\dfrac{1}{6}\Big[j_1\left(l+b\right)+ j_2\left(l+a\right)\Big]k$	$\dfrac{1}{2}jkl$
◡ k, l	$\dfrac{5}{12}jkl$	$\dfrac{1}{4}jkl$	$\dfrac{1}{12}\left(3j_1+5j_2\right)kl$	$\dfrac{2}{3}jkl$
◠ k, l	$\dfrac{1}{4}jkl$	$\dfrac{5}{12}jkl$	$\dfrac{1}{12}\left(5j_1+3j_2\right)kl$	$\dfrac{2}{3}jkl$
◞ k, l	$\dfrac{1}{4}jkl$	$\dfrac{1}{12}jkl$	$\dfrac{1}{12}\left(j_1+3j_2\right)kl$	$\dfrac{1}{3}jkl$
k ◜, l	$\dfrac{1}{12}jkl$	$\dfrac{1}{4}jkl$	$\dfrac{1}{12}\left(3j_1+j_2\right)kl$	$\dfrac{1}{3}jkl$
⌢ k, l	$\dfrac{1}{3}jkl$	$\dfrac{1}{3}jkl$	$\dfrac{1}{3}\left(j_1+j_2\right)kl$	$\dfrac{2}{3}jkl$

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